Two-User Gaussian Interference Channels: An Information Theoretic Point of View

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Abstract

The purpose of this monograph is to introduce up-to-date capacity theorems for the two-user Gaussian interference channel, including both single-antenna and multiple-antenna cases.

The monograph starts with the single antenna case and introduces the Han and Kobayashi achievable rate region and its various subregions. Several capacity outer bounds are then presented; these outer bounds, together with the achievable rate region, yield several capacity results for the single-antenna Gaussian interference channel. They include the capacity region for strong interference and the sum-rate capacity for Z interference, noisy interference, mixed interference, and degraded interference.

For the more complex multiple-antenna case, the interference state is no longer determined solely by the interference strength, as is the case for the single-antenna Gaussian interference channel. Instead, the structure of the interference in different multi-dimensional subspaces plays an equally important role. As a result of this multiple-dimensional signaling, new interference states, including generally strong, generally noisy, and generally mixed interference, are introduced to obtain capacity theorems that generalize those for the single-antenna case.

Notations and Acronyms

 Scalars, vectors and matrices

\( X \)  scalar \( X \)

\( x \)  vector \( x \)

\( X \)  matrices \( X \)

\( x^n \)  \( x^n = \{x_1, x_2, \ldots, x_n\} \), a sequence of vectors \( x_i \),
          \( i = 1, \ldots, n \)

\( I \)  identity matrix

\( I_{(k)} \)  \( k \times k \) identity matrix

\( 0 \)  all-zero vector or matrix

\( 0_{(k)} \)  \( k \times k \) all-zero matrix

\( 0_{k \times r} \)  a \( k \times r \) all-zero matrix

\( x^T \) or \( X^T \)  transpose of vector \( x \) or matrix \( X \)

\( x^H \) or \( X^H \)  conjugate transpose of vector \( x \) or matrix \( X \)

\( X^{-1} \)  inverse of matrix \( X \)

\( \|x\| \)  Euclidean vector norm of \( x \), i.e., \( \|x\|^2 = x^T x \)

\( |X| \)  determinant of matrix \( X \)

\( \text{tr}(X) \)  trace of matrix \( X \)

\( \text{rank}(X) \)  rank of matrix \( X \)

\( \text{Vec}(X) \)  vectorization of matrix \( X \)

\( \text{radius}(X) \)  numerical radius of matrix \( X \)

\( A \otimes B \)  Kronecker product of matrices \( A \) and \( B \)

\( A \succeq B \)  matrices \( A \), \( B \), and \( A - B \) are all symmetric and
Notations and Acronyms

probability density function of random variable $X$
conditional density function of $X$ given $Y$
$x$ has a Gaussian distribution with zero mean and covariance matrix $\Sigma$
eX expectation of random variable $X$
variance of random variable $X$
covariance matrix of random vector $x$
cross covariance matrix of random vectors $x$ and $y$
entropy of discrete random variable $X$
differential entropy of random variable $X$
mutual information between $X$ and $Y$
absolute value of $X$
arctangent function
logarithm of $x$
sign function

positive semi-definite.
Interference commonly exists in multi-user wireless networks in which all the users share the same communication medium. While sending information to its intended receiver, each transmitter generates interference to all other receivers. The existence of interference may heavily degrade the overall performance of the system. Therefore, each user has to choose prudently its transmission strategy in order to optimize the performance of the entire network. Substantial effort has been devoted to understanding the impact of interference and to looking for capacity-achieving schemes of dealing with interference.

The scope of this monograph is limited to the study of a two-user Gaussian interference channel with a static setting, i.e., the channel state is assumed fixed and known to both the transmitters and the receivers. There have been significant efforts in studying more complex interference channel models, including that of multiple users and/or with fading channels. Our focus on the simple two-user Gaussian interference channel allows us to explore some of the subtle yet potentially fundamental aspects of interference in more details. It is not unreasonable to state that our understanding for such a simple channel model is still rather limited especially in some important parameter regimes.
1.1 Interference Channel Model

The two-user interference channel (IC) was first introduced by Shannon [55]. This model consists of two pairs of transceivers, in which each transmitter communicates with its intended receiver while generating interference at the other receiver. The channel model and its related encoding decoding functions are defined as follows, for user $i$, $i = 1, 2$:

- **Message set**: $\mathcal{M}_i = \{1, 2, \ldots, 2^{nR_i}\}$
- **Encoding function**: $E_{n_i}: k_i \mapsto X_i^n(k_i), \quad k_i \in \mathcal{M}_i$
- **Channel statistics**: $p(y_1^n y_2^n | x_1^n x_2^n) = \prod_{j=1}^n p(y_{1j} y_{2j} | x_{1j} x_{2j})$
- **Decoding function**: $D_{e_i}: Y_i^n \rightarrow \hat{k}_i$
- **Error probability**: \[
\frac{1}{2^{n(R_1+R_2)}} \sum_{k_1=1}^{2^{nR_1}} \sum_{k_2=1}^{2^{nR_2}} \Pr\left\{ \left(\hat{k}_1, \hat{k}_2\right) \neq (k_1, k_2) \mid (k_1, k_2) \text{ sent} \right\}
\]

where $n$ is the block length, $X_i^n = \{X_{i1}, \ldots, X_{in}\}$ and $Y_i^n = \{Y_{i1}, \ldots, Y_{in}\}$ are respectively the transmitted and received signal sequences, $k_i$ is the message index and $\hat{k}_i$ is its estimate, and $R_i$ is the transmission rate.

If for a given $(R_1, R_2)$ pair, there exist encoding and decoding functions such that the error probability is arbitrarily small when $n \rightarrow \infty$, then the rate pair $(R_1, R_2)$ is achievable. The closure of the collection of all achievable rate pairs is the capacity region of this IC.

1.1.1 Capacity and Channel Transition Probabilities

Though the channel statistics of an IC is characterized by joint channel transition probability $p(y_1 y_2 | x_1 x_2)$, the capacity is often determined by marginal channel transition probabilities $p(y_1 | x_1 x_2)$ and $p(y_2 | x_1 x_2)$. The reason is that decoding is performed individually at each receiver. Therefore, dependence between $Y_1$ and $Y_2$ for a given pair of $X_1$ and $X_2$...
does not have impact on the decoding performance. For a GIC, dependence between \( Y_1 \) and \( Y_2 \) given \( X_1 \) and \( X_2 \) can be introduced through, and controlled by, the correlation between the additive noises at the two receivers. Any two GICs with different transition probabilities still have the same capacity as long as their respective marginal transition probabilities are identical.

### 1.1.2 Single-Antenna Gaussian Interference Channel

Specializing to the Gaussian case, the received signals can be written as

\[
\begin{align*}
Y_1 &= h_1 X_1 + f_2 X_2 + \sigma_{z1} Z_1 \\
Y_2 &= h_2 X_2 + f_1 X_1 + \sigma_{z2} Z_2
\end{align*}
\]

(1.1a, 1.1b)

where for \( i = 1, 2 \), \( h_i \) and \( f_i \) are channel coefficients known at both transmitters and receivers, \( Z_i \) is zero-mean Gaussian noise with unit variance, \( \sigma_{zi} \) is a positive constant, and the transmitted signal \( X_i \) is subject to an average power constraint:

\[
\sum_{j=1}^{n} E \left( X_{ij}^2 \right) \leq n \bar{P}_i.
\]

(1.2)

It is easy to show that the capacity region of the Gaussian interference channel (GIC) defined in (1.1) is equivalent to that of the GIC in the standard form [11]:

\[
\begin{align*}
Y_1 &= X_1 + a_2 X_2 + Z_1 \\
Y_2 &= X_2 + a_1 X_1 + Z_2
\end{align*}
\]

(1.3a, 1.3b)

where

\[
\begin{align*}
a_1 &= \frac{f_2 \sigma_{z2}}{h_2 \sigma_{z1}} \\
a_2 &= \frac{f_1 \sigma_{z1}}{h_1 \sigma_{z2}} \\
\bar{P}_i &= \frac{h_i^2}{\sigma_{zi}^2} \bar{P}_i.
\end{align*}
\]

(1.4a, 1.4b, 1.4c)

The GIC in the standard form is shown in Figure 1.1 where the transmit signals are subject to respective power constraints \( P_1 \) and \( P_2 \).
1.2 Existing Results for Gaussian Interference Channels

The IC model was first introduced by Shannon [55]. Alshwede [2] derived a limit expression for the capacity region:

\[
\bigcup \left\{ \langle R_1, R_2 \rangle \mid R_1 \leq \lim_{n \to \infty} \frac{1}{n} I(X_1^n; Y_1^n) \right\} = \bigcup \left\{ \langle R_1, R_2 \rangle \mid R_2 \leq \lim_{n \to \infty} \frac{1}{n} I(X_2^n; Y_2^n) \right\}.
\]

(1.5)

However, the practical significance of the above limit expression for the capacity region is not clear. Attempting to directly evaluate the limit expression of channel capacity is known to be a fruitless exercise [13]. Even for the simple case of the Gaussian multiple access channel, calculating the capacity using the limit expression was shown to yield a rate region that is strictly smaller than the capacity region when the input is limited to multivariate Gaussian distributions. This is despite the fact that Gaussian input achieves the single-letter expressed capacity region of the Gaussian multiple access channel. For two-user GIC, except for the noisy-interference case, multivariate Gaussian input distribution was also shown to be sum-rate sub-optimal using the limit expressions [1], even though Gaussian input distribution indeed achieves the sum-rate capacity for strong interference. As such, one can not expect that the limit expression for the GIC capacity region collapses into a single-letter expression that is easy to evaluate, and provides insight to capacity achieving coding schemes.

Many efforts have thus been devoted to the design of various coding
Introduction

schemes that lead to achievable regions with single-letter expressions. The best inner bound was obtained by Han and Kobayashi [24] using superposition encoding and joint decoding. This region was later simplified by Chong, Motani, Garg and El Gamal in [14] and by Kramer in [29]. Early outer bounds on the capacity region can be found in [37], [38], [12]. Cooperation and genie-aided outer bounds can be found in [28], [22], [51], [34], [3]. Etkin, Tse, and Wang showed that the Han and Kobayashi achievable region is within a half bit per user of the capacity region [22].

The first capacity region of the GIC was obtained by Carleial in [10] for the very strong interference case, in which the capacity is achieved by decoding and subtracting interference before decoding the useful signals. This result was extended to the strong interference case in [24] and [39], in which receivers jointly decode the interference and the useful signal to achieve capacity. The sum-rate capacity of the degraded GIC was obtained in [38]. It was shown in [15] that the capacity region of a Gaussian Z interference channel (GZIC) is equivalent to that of a degraded GIC. Therefore, the sum-rate capacity of a GZIC is obtained directly from [38] (see [36, Theorem 2]). Recently, it was shown in [51], [34], and [3] that the sum-rate capacity is achieved by treating interference as noise if the GIC satisfies a simple condition. This kind of GIC is said to have noisy interference. The sum-rate capacity for GICs with mixed interference was determined in [34] and [61].

On the study of the capacity for the multiple-input multiple-output (MIMO) GIC and its two special cases, namely, the multiple-input single-output (MISO) GIC and single-input multiple-output (SIMO) GIC, Telatar and Tse [58] showed that the Han-Kobayashi region is within a half bit per user and per receive antenna of the capacity region. Vishwanath and Jafar [60] determined the capacity region for the SIMO GIC with strong interference. Recent work in [45] and [47] derived the capacity region of a MIMO GIC for the strong interference case, and the sum-rate capacity for strong Z interference, weak Z interference, noisy interference and mixed interference. Other noisy interference conditions were obtained in [4] for MIMO GIC with full-rank optimal input covariance matrices and symmetric MISO and SIMO
GICs, and in [48] for parallel GICs (diagonal channel matrices). The noisy interference sum-rate capacity in [45], [47], [4], and [48] was later generalized in [54]. Realizing the fact that very strong interference is not necessarily a special case of strong interference for a MIMO GIC (a sharp contrast to the single-antenna GIC) [47], [46], a new category of interference type, referred to as generally strong interference, which includes both strong and very strong interferences as special cases, was introduced in [53]. The capacity region of the MIMO GIC with generally strong interference is achieved by jointly decoding the signal and interference. This new generally strong interference condition can determine the capacity region of MIMO GICs that cannot be determined by the traditional strong interference condition (e.g., MISO GICs).

1.3 Outline of Monograph

In this chapter, we have introduced the system model and the standard form for the single-antenna GIC. A brief summary of existing capacity results for GIC has also been given.

In Chapter 2, we present the Han-Kobayashi (H-K) achievable rate region and some of its subregions. Several important results, including the role of time sharing, the equivalence in capacity region between the GZIC and the degraded GIC, the so-called ‘noiseberg’ approach for the GZICs [31], are discussed in details.

In Chapter 3, we introduce several outer bounds on the capacity region. These outer bounds lead to the bounded gap between the H-K region and the capacity region, as well as various capacity results, including the capacity region for strong and very strong interference, the sum-rate capacity for noisy interference, Z interference, and mixed interference.

In Chapter 4, we introduce the standard form channel model for MIMO GICs and an effective way to compute the H-K region for MISO GICs. Subsequently, we extend all the capacity results for single-antenna GICs to multiple-antenna GICs. These include the capacity region for strong interference and very strong interference, and the sum-rate capacity for noisy interference. In addition, we introduce a
new concept called generally strong interference, which generalizes the notion of strong interference, and obtain the capacity region under such conditions. The generally strong interference also allows us to obtain the sum-rate capacity for mixed interference. We also determine part of the capacity boundary for the MIMO GIC with weak Z interference and mixed interference.
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