A Fresh Look at Generalized Sampling

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## Contents

1 Introduction 2

2 Background 8
   2.1 Reconstruction kernels 8
   2.2 Analysis filters 10

3 Basic notation, definitions, and properties 12

4 Fundamental algorithms 18
   4.1 Interpolation 19
   4.2 Inverse discrete convolution 20
   4.3 Orthogonal projection 22
   4.4 Oblique projection 25

5 Translation and scaling 27
   5.1 Translation of discretized signals 27
   5.2 Scaling of discretized signals 29

6 Approximation of derivatives 37

7 Generalized prefiltering and estimator variance 41

8 Practical considerations 47
8.1 Grid structure ..................................................... 47
8.2 Efficient use of piecewise-polynomial kernels ............... 48
8.3 Prefiltering, reconstruction, and color spaces ................ 50
8.4 Range constraints .................................................. 51

9 Theoretical considerations ........................................ 53

10 Experiments and analyses ........................................ 57

11 Conclusions .......................................................... 68

Acknowledgements .................................................. 69

Appendices ............................................................. 70

A Source-code ......................................................... 71

Bibliography ........................................................... 81
Abstract

Discretization and reconstruction are fundamental operations in computer graphics, enabling the conversion between sampled and continuous representations. Major advances in signal-processing research have shown that these operations can often be performed more efficiently by decomposing a filter into two parts: a compactly supported continuous-domain function and a digital filter. This strategy of “generalized sampling” has appeared in a few graphics papers, but is largely unexplored in our community. This survey broadly summarizes the key aspects of the framework, and delves into specific applications in graphics. Using new notation, we concisely present and extend several key techniques. In addition, we demonstrate benefits for prefiltering in image downscaling and supersample-based rendering, and analyze the effect that generalized sampling has on the noise due to Monte Carlo estimation. We conclude with a qualitative and quantitative comparison of traditional and generalized filters.
Many topics in computer graphics involve digital processing of continuous-domain data, so it is unsurprising that discretization and reconstruction are essential operations. Figure 1.1 shows the traditional sampling and reconstruction pipeline. During discretization (e.g., rasterization of a scene, or capture of a digital photograph), a continuous input signal $f$ is passed through an

**Figure 1.1:** The traditional signal-processing pipeline is divided into two major stages: discretization and reconstruction. During discretization, a continuous input signal $f$ is convolved with the reflection $\psi^r$ of a given analysis filter $\psi$. The resulting prefiltered signal $f_\psi = f * \psi^r$ is then uniformly sampled into a discrete sequence $[f_\psi]$. To obtain the output approximation $\tilde{f}$, the reconstruction stage computes the mixed convolution between $[f_\psi]$ and a given reconstruction kernel $\varphi$, i.e., a sum of shifted copies of $\varphi$, where each shifted copy scaled by the corresponding entry in $[f_\psi]$. (Our notation is explained in greater depth in section 3.)
**Prefiltering** \[ f \star \psi = \beta^0 \]

**Sampling** \[ f\psi \times \text{III} = [f\psi] \]

**Reconstruction** \[ Jf\psi \star \varphi = \beta^1 = \tilde{f} \]

**Figure 1.2:** A continuous function \( f \) is prefiltered with analysis kernel \( \psi \) (here the box function \( \beta^0 \), not to scale). The resulting signal \( f\psi \) is sampled into a discrete sequence \([f\psi]\). The final output \( \tilde{f} \) is obtained by mixed convolution between the discrete sequence \([f\psi]\) and the reconstruction kernel \( \varphi \) (here the hat function \( \beta^1 \), not to scale).

*analysis filter* \( \psi \) (a.k.a. *sampling kernel*, *prefilter*, or *antialiasing filter*) before being sampled. The result is a discrete sequence \([f\psi]\) (e.g., an image). During reconstruction (e.g., interpolation of a texture, or display of an image on a screen), the continuous approximation \( \tilde{f} \) of the original signal is obtained by mixed convolution with a *reconstruction kernel* \( \varphi \) (a.k.a. *generating function*, *basis function*, or *postfilter*). Figure 1.2 illustrates each step of the process with a concrete example in 1D.

The roles of the analysis filter \( \psi \) and reconstruction kernel \( \varphi \) are traditionally guided by the sampling theorem [Shannon, 1949]. Given a sampling rate \( 1/T \), the analysis filter \( \psi = \text{sinc}(\cdot/\tau) \) eliminates from the input signal \( f \) those frequencies higher than or equal to \( 1/2T \) so that the bandlimited \( f\psi \) can be sampled without aliasing. And in that case, the reconstruction kernel \( \varphi = \text{sinc}(\cdot/\tau) \) recreates \( \tilde{f} = f\psi \) exactly from the samples.

Sampling may also be interpreted as the problem of finding the function \( \tilde{f} \) that minimizes the norm of the residual \( \|f - \tilde{f}\|_{L^2} \). If we restrict our attention
Introduction

The main idea in generalized sampling is to broaden the analysis and reconstruction kernels by expressing these as mixed convolutions \((p \ast \psi)\) and \((r \ast \varphi)\) with a pair of digital filters \((p\) and \(r\)) while retaining compact support for the functions \(\psi\) and \(\varphi\).

Figure 1.3: The main idea in generalized sampling is to broaden the analysis and reconstruction kernels by expressing these as mixed convolutions \((p \ast \psi)\) and \((r \ast \varphi)\) with a pair of digital filters \((p\) and \(r\)) while retaining compact support for the functions \(\psi\) and \(\varphi\).

Figure 1.4: The traditional Keys cubic (Catmull-Rom spline) \(K\) has support 4 and a reasonably sharp frequency response \(|\hat{K}|\). The cardinal cubic B-Spline \(\beta^3\) int is a generalized kernel formed from the basic cubic B-spline \(\beta^3\) and a digital filter. The digital filter acts to widen support to infinity (though with exponential decay) and to significantly sharpen the frequency response.

to the space of bandlimited functions, the ideal prefilter is still \(\psi = \text{sinc}(\cdot/T)\). However, functions are often not bandlimited in practice (e.g., due to object silhouettes, shadow boundaries, vector outlines, detailed textures), and for efficiency we desire \(\psi\) and \(\varphi\) to be compactly supported.

In addressing these concerns, the signal-processing community has adopted a generalization of the sampling and reconstruction pipeline [Unser 2000]. The idea is to represent the prefilter and reconstruction kernels as mixed convolutions of compactly supported kernels and digital filters. As shown in figure 1.3 digital filters \(p\) and \(r\) respectively modify the prefilter \(\psi\) and the reconstruction kernel \(\varphi\). The additional degrees of freedom and effectively larger filter support enabled by \(p\) and \(r\) allow the design of generalized kernels with better approximation properties or sharper frequency response. Figure 1.4 compares a traditional piecewise cubic kernel (the Catmull-Rom spline, or Keys cubic) with a generalized cubic kernel (the cardinal cubic B-spline).
Figure 1.5: Generalized sampling adds a digital filtering stage to the pipeline. The output $[f_{\psi}]$ of the sampling stage is convolved with a digital transformation filter $q = p^* r$. It is the result $c$ of this stage (and not $[f_{\psi}]$) that is convolved with the reconstruction kernel $\varphi$ to produce the output signal.

Equivalently, the digital filters $p$ and $r$ can be combined as $q = p^* r$ into a separate filter stage as shown in figure 1.5. The result $[f_{\psi}]$ of the sampling stage is transformed by the digital filter $q$ (a.k.a. correction or basis change) to form a new discrete sequence $c$, which is then convolved with $\varphi$ as usual to reconstruct $\tilde{f}$. The key to the efficiency of this generalized sampling framework is that the digital filters $p$ and $r$ that arise in practice are typically compact filters or their inverses [Unser et al., 1991], both of which are parallelizable on multicore CPU and GPU architectures [Ruijters et al., 2008, Nehab et al., 2011]. Thus, the correction stage adds negligible cost.

An important motivation for generalized sampling is improved interpolation [Blu et al., 1999]. As demonstrated in figure 1.6, an image $[f_{\psi}]$ is processed by a digital filter $q$ resulting in a coefficient array $c$ which can then be efficiently reconstructed with a simple cubic B-spline filter $\beta^3$. The resulting interpolation is sharper and more isotropic (i.e., has higher quality) than that produced by the popular Mitchell-Netravali filter [1988], even though both filters have the same degree and support. The implementation of the digital filtering stage is described in detail in section 4.2. The theory of image upscaling is described in section 5.2 with implementation notes in section 8.2. Source-code is provided in appendix A.

In graphics, careful prefiltering is often necessary to prevent aliasing. McCool [1995] describes an early application of generalized sampling, in which rendered triangles are antialiased analytically by evaluating a prism spline prefilter. The resulting image is then convolved with a digital filter. In this work, we apply generalized sampling to image downscaling and in general
Introduction

Input $f$  
\[ \rightarrow \]
Reconstructed $f \ast M$  

Corrected $c = f \ast r$  
\[ \rightarrow \]
Reconstructed $c \ast \beta^3$

Figure 1.6: Reconstruction example. The top row shows the result of the traditional cubic Mitchell-Netravali filter $M$. The bottom row uses the generalized sampling approach, first applying a digital filter $r = \left[ \frac{1}{8}, \frac{1}{8}, \frac{1}{8} \right]$ as a preprocess, and then reconstructing with the cubic B-spline $\beta^3$ — which is less expensive to evaluate on a GPU than filter $M$. To rendering with supersampling. Figure 1.7 shows an example. The input $f$ is prefiltered using the cubic B-spline basis $\beta^3$. The resulting over-blurred image is then transformed with a digital filter $p'$ that reshapes the antialiasing kernel \textit{a posteriori}. The final low-resolution image is sharper and exhibits less aliasing than with a Catmull-Rom filter, for a similar computational cost. The theory of image downscaling is described in section 5.2, with implementation notes in section 8.2 and source-code in appendix A. Generalized supersampling is described in section 7.

Our aim is to present a concise overview of the major developments in generalized sampling and to extend these techniques to prefiltering in graphics. To facilitate exposition and exploration, we develop a new concise notation for sampling. With this parameter-free notation, key techniques are derived using simple algebraic manipulation. We conclude by comparing a variety of
Figure 1.7: Prefiltering example. The top row shows the result of rendering with the Keys (Catmull-Rom) prefilter $K$. The bottom row shows rendering using a B-spline $\beta^3$, followed by convolution with a digital filter $p^\gamma = \left[\frac{1}{6}, \frac{1}{3}, \frac{1}{6}\right]^\gamma$. The generalized prefilter $p * \beta^3$ equals the cubic cardinal B-spline $\beta^3_{\text{int}}$. Kernels $K$ and $\beta^3$ have the same support, but the improved frequency response of $\beta^3_{\text{int}}$ reduces aliasing while maintaining sharpness. (Our notation is explained in section 3.)

traditional and generalized filters, using frequency-domain visualizations and empirical experiments using both $L_2$ and SSIM metrics, to identify the best strategies available.


Bibliography


