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Asymptotic Frame Theory for Analog Coding

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Full text available at: http://dx.doi.org/10.1561/0100000125

Foundations and Trends® in Communications and Information Theory

Published, sold and distributed by:
now Publishers Inc.
PO Box 1024
Hanover, MA 02339
United States
Tel. +1-781-985-4510
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Outside North America:
now Publishers Inc.
PO Box 179
2600 AD Delft
The Netherlands
Tel. +31-6-51115274

The preferred citation for this publication is

M. Haikin, M. Gavish, D.G. Mixon and R. Zamir. *Asymptotic Frame Theory for Analog Coding*. Foundations and Trends® in Communications and Information Theory, vol. 18, no. 4, pp. 526–645, 2021.

ISBN: 978-1-68083-909-8
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Volume 18, Issue 4, 2021

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Asymptotic Frame Theory for Analog Coding

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ABSTRACT

Over-complete systems of vectors, or in short, frames, play the role of analog codes in many areas of communication and signal processing. To name a few, spreading sequences for code-division multiple access (CDMA), over-complete representations for multiple-description (MD) source coding, space-time codes, sensing matrices for compressed sensing (CS), and more recently, codes for unreliable distributed computation. In this survey paper we observe an information-theoretic random-like behavior of frame subsets. Such sub-frames arise in setups involving erasures (communication), random user activity (multiple access), or sparsity (signal processing), in addition to channel or quantization noise. The goodness of a frame as an analog code is a function of the eigenvalues of a sub-frame, averaged over all sub-frames (e.g., harmonic mean of the eigenvalues relates to least-square estimation error, while geometric mean to the

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Marina Haikin, Matan Gavish, Dustin G. Mixon and Ram Zamir (2021), "Asymptotic Frame Theory for Analog Coding", Foundations and Trends® in Communications and Information Theory: Vol. 18, No. 4, pp 526–645. DOI: 10.1561/0100000125.
©2021 M. Haikin, M. Gavish, D.G. Mixon and R. Zamir

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Shannon transform, and condition number to the restricted isometry property).

Within the highly symmetric class of Equiangular Tight Frames (ETF), as well as other "near ETF" families, we show a universal behavior of the empirical eigenvalue distribution (ESD) of a randomly-selected sub-frame: (*i*) the ESD is asymptotically indistinguishable from Wachter's MANOVA distribution; and (*ii*) it exhibits a convergence rate to this limit that is indistinguishable from that of a matrix sequence drawn from MANOVA (Jacobi) ensembles of corresponding dimensions. Some of these results follow from careful statistical analysis of empirical evidence, and some are proved analytically using random matrix theory arguments of independent interest. The goodness measures of the MANOVA limit distribution are better, in a concrete formal sense, than those of the Marchenko–Pastur distribution at the same aspect ratio, implying that deterministic analog codes are better than random (i.i.d.) analog codes. We further give evidence that the ETF (and near ETF) family is in fact superior to any other frame family in terms of its typical sub-frame goodness.

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1

Introduction

A frame is an "over-complete basis", i.e., a system of vectors that spans the space with more vectors than the space dimension (real or complex). Let us denote the space dimension by m, and the number of vectors by n, where $n > m$. The $m \times n$ frame matrix

$$F = [\mathbf{f}_1 \cdots \mathbf{f}_n] \tag{1.1}$$

is generated by stacking the frame vectors $\mathbf{f}_1, \ldots, \mathbf{f}_n$ as columns, where we restrict attention to unit-norm vectors $\|\mathbf{f}_i\| = 1$. The relative position of frame vectors is determined by their pairwise cross-correlation matrix

$$F^\dagger F = \Big\{\langle \mathbf{f}_i, \mathbf{f}_j \rangle\Big\}, \tag{1.2}$$

called also Gram or covariance matrix, which is invariant under unitary operation on the frame vectors (e.g., rotation).

This survey proposes an information-theoretic view on the design and analysis of frames with favorable performance. We think of a frame as an "analog code", which can add redundancy [66], [90], [128], [151], remove redundancy [22], or multiplex information directly in the signal space [99], [120], [153]. Multiplication by the frame matrix can expand

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the dimension $m : n$ hence add redundancy, or reduce the dimension $n : m$ hence compress (with $\mathbf{y} = F^{\dagger}\mathbf{x}$ for the former and $\mathbf{x} = F\mathbf{y}$ for the latter). The aspect ratio n/m is often called the "frame redundancy".

Although information theory tells us that reliable data transmission (adding redundancy) and compression (removing redundancy) can be achieved by *digital* codes, real-world physical-layer communication systems combine *analog* modulation techniques that can be described in terms of frames[1] [16], [17], [22], [67], [94], [158]. We are specifically interested in some old and new applications of frames that involve a combination of *random activity* and *noise*. Performance in these applications is a function of the eigenvalues of (the Gram of) a randomly selected k-subset F_S of the n frame vectors,

$$F_S = [\mathbf{f}_{i_1} \cdots \mathbf{f}_{i_k}] \tag{1.3}$$

where $S = \{i_1, \ldots, i_k\} \subset [n]$, $|S| = k$. (We shall use $[n]$ to denote $\{1, \ldots, n\}$.)

For example, in *non-orthogonal code-division multiple access* (NOMA-CDMA), [127], [140], [145], n users are allocated with m-length spreading sequences using the frame F, but only k out of the n users are active at any given moment, and performance is measured by the *Shannon capacity* of the vector Gaussian channel associated with the $m \times k$ sub-matrix F_S (averaged over the subset S of active users). In transform-based *multiple-description* (MD) source coding, [67], [110], an m-dimensional vector source is expanded into n packets using the frame F, only k packets are received, and performance is measured by the *remote rate-distortion function* associated with F_S (averaged over the subset S of received packets). In coded distributed computation (CDC), [47], [85], [89], the user (master) node expands m sub-computation tasks into n redundant tasks using the frame F, and sends them to n *noisy* computation nodes (where the noise is due to finite precision computation); only k nodes return their answers on time ($n - k$ are stragglers), and

[1] For example, coded-modulation can be thought of as concatenation of an outer digital code with an inner analog code.

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the sub-matrix F_S determines the final average precision (or noise amplification) after the user node decodes the desired computation value. Other "(n, m, k) setups" are listed in Table 1.1.

Table 1.1: (n, m, k) setups featuring analog frame codes

Application	n	m	k	$m \gtrless k$	References
Source with erasures	block-length	bandwidth	important samples	$m > k$	[72], [153]
NOMA-CDMA	users	resources (spread)	active users	$m > k$	[120], [128], [158]
Impulsive channel	block-length	bandwidth	non-erased	$m < k$	[138], [151]
Space-time coding	space (diversity)	time	non-erased antennas	$m < k$	[131], [137], Section 9.3
$\Delta\Sigma$ modulation	over-sampled	original	non-erased	$m < k$	[25], [64]
Multiple descriptions	transmitted	original	received	$m < k$	[67], [110]
Wavelets	coefficients	source	significant coefficients	$m > k$	[81]
Compressed sensing	input	output	sparsity	$m > k$	[24], [40]
Coded computation	workers	computations	non-stragglers	$m < k$	[47], [85], [89]
Neural networks	input	output	features	$m > k$	[7]

In an ideal *noiseless* setup one could choose the frame redundancy n/m equal to the reciprocal of the activity ratio k/n, i.e., $m = k =$ the effective number of users/packets/nodes in the examples above. This is similar to digital erasure correction using *maximum distance separable* (MDS) codes for a channel with k out of n non-erased symbols [13]. However, when *noise* is involved (channel/quantization/computation noise in the setups above), a better trade-off between noise immunity and information rate is obtained by choosing a lower/higher frame redundancy; $k < m < n$ in NOMA-CDMA, or $m < k < n$ in MD and CDC. Thus, m is a design parameter that we can optimize.

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Frame design could be viewed as an attempt to find vectors $\mathbf{f}_1, \ldots, \mathbf{f}_n$ in $\mathbb{R}^m$ or $\mathbb{C}^m$, $n > m$, that are somehow "as orthogonal to each other as possible", either in pairs ($k = 2$) or in larger k-subsets [81], [148]. As we shall see in Chapter 3, the sub-frame performance criteria mentioned above (capacity, rate-distortion function, noise amplification), denoted in general as $\Psi(F_S)$, depend on the spread of the eigenvalues of the Hessian $F_S F_S^\dagger$ or Gram $F_S^\dagger F_S$ matrices[2] of the sub-frame F_S. More mutual orthogonality amounts to a more compact eigenvalue spectrum, and ideal performance occurs when the spectrum shrinks to a delta function, or equivalently, the sub-frame is orthogonal. The redundant nature of the frame, however, implies that most of its subsets are *not* orthogonal. Our target is therefore to find a frame whose *average* performance over all k-subsets

$$\bar{\Psi}(F, k) = \frac{1}{\binom{n}{k}} \sum_{\substack{S \subset [n] \\ |S| = k}} \Psi(F_S) \tag{1.4}$$

is "good"; or in other words, a frame whose *typical* subset has a *compact* eigenvalue spectrum.[3]

We borrow from information theory the probabilistic view of a communication channel, and the notions of typicality and typical-case (rather than worst-case) goodness [34]. The information-theoretic viewpoint leads us to look for frames with a "typically compact" subset spectrum for a given (n, m, k) triplet, and for frame *families* with the best attainable *asymptotic* goodness in the limit as n goes to infinity for fixed (asymptotic) redundancy ratios n/m and k/m. This fresh look on frames turns out to be fruitful, and opens many interesting questions at the intersection of signal processing, random matrix theory, geometry, harmonic analysis and information theory.

Sampling theory suggests the *low-pass frame* (LPF), the frame analog of band-limited interpolation, as a practical candidate for signal

[2] The nonzero eigenvalues of both matrices are the same.

[3] Simple (though only partial) measures for spectrum compactness are the variance and kurtosis; see Section 10.2.

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expansion. This turns out to be a far-from-optimal choice, as we shall see, due to large noise amplification for a typical subset (which corresponds to noisy reconstruction from a non-uniform sampling pattern [83], [98], [122], [144]).

Information theory suggests *random* (i.i.d.) frames as natural candidates for good analog codes. To study the spectrum of these objects, *Random Matrix Theory* (RMT) offers a helpful matrix version of the law of large numbers: the eigenvalue distribution of a typical random matrix tends to *concentrate* towards a fixed distribution in the limit of large dimensions [1], [49]. Indeed, if we choose the elements of the frame matrix F as i.i.d. Gaussian variables, then the subset Gram matrix $F_S^\dagger F_S$ is drawn from a Wishart ensemble, and its spectrum converges almost surely in distribution to the Marchenko–Pastur (MP) distribution with parameter $\beta = k/m$ [93]. We can thus compute the capacity / rate-distortion function / noise amplification (1.4) associated with the Marchenko–Pastur distribution, and obtain some achievable asymptotic performance $\Psi(\mathrm{MP}, \beta)$ for the problems described above.

Is the Marchenko–Pastur distribution - corresponding to random i.i.d. frames - the “most compact” subset spectrum we can hope for? One of the key results of this survey is that better *deterministic* frames do exist. In fact, a certain class of highly symmetric frames obeys asymptotic concentration of the spectrum of a randomly-selected subset to a universal limiting distribution – similarly to the case of a completely random (i.i.d.) matrix. Crucially, this limiting distribution is *more compact* than the MP distribution.

Equiangular tight frames (ETF) are in a sense the most geometrically symmetric family of frames [26], [54], [148]. They have numerous applications in communications and signal analysis, [90], and their study brings together geometry, combinatorics, probability, and harmonic analysis. Interestingly, as we shall see in Chapter 4, one construction of ETFs corresponds to signal expansion with an *irregular* Fourier transform, [153], as opposed to the low-pass frame (LPF) mentioned above.

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A series of recent papers [71]–[75], [92], demonstrated that ETFs, as well as other deterministic tight frames that we term "near ETFs", exhibit an RMT-like behavior familiar from Multivariate ANalysis Of VAriance (MANOVA) [46], [106], [146]. We shall call this phenomenon the *ETF-MANOVA relation*.

Specifically, the work in [74] showed empirically that for any frame within this class of frames, the eigenvalue distribution of a randomly-selected subset appears to be indistinguishable from that of a random matrix taken from the MANOVA (Jacobi) ensemble. The work in [71], [73], [75], [92] further partially proved analytically[4] that as $n \to \infty$, for aspect ratios $m/n \to \gamma$ and $k/m \to \beta$, this eigenvalue distribution converges in distribution almost surely to Wachter's limiting MANOVA distribution parameterized by γ and β [146]. The concluded asymptotic performance (1.4) of the ETF family,

$$\bar{\Psi}(\mathrm{ETF}, \gamma, \beta) = \Psi(\mathrm{MANOVA}, \gamma, \beta), \tag{1.5}$$

is strictly better than $\Psi(\mathrm{MP}, \beta)$, the asymptotic performance of random (i.i.d.) frames, for various performance measures Ψ. Figure 1.1 shows that the gain of MANOVA over MP is ~ 2 dB in noise amplification, and ~ 0.35 bit in capacity. We conjecture that in terms of these performance measures, the MANOVA distribution is, in fact, the *most compact* typical sub-frame spectrum achievable by *any* unit-norm frame.

In this survey paper we propose a common framework for these topics, located at the intersection of information theory and neighboring fields. Chapter 2 addresses primarily the information theory audience, and motivates a passage from digital codes to low-pass interpolation and analog frame codes, through a side-information source coding problem. Chapter 3 formalizes the notion of a performance measure

[4] The proof is complete for the case $\gamma = 1/2$ [92]. For a general $0 < \gamma < 1$, [73] establishes a recursive formula in r for the asymptotic mean rth moment of a randomly-selected ETF subset, for $r = 1, 2, \ldots$. Using a symbolic computer program, we were able to verify that this formula coincides with the first 10 MANOVA moments (above which the complexity explodes). A proof of the identity for a general $r \in \mathbb{N}$ remains a fascinating open problem.

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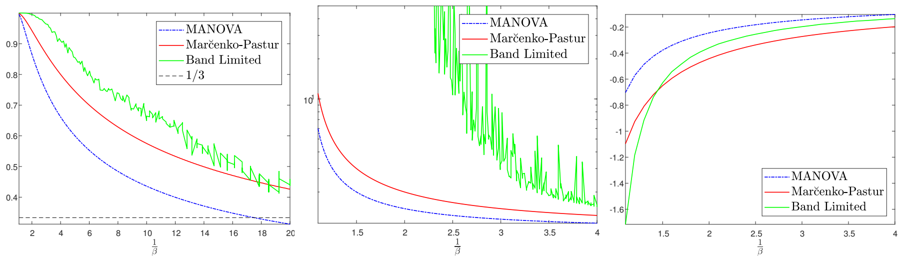

Figure 1.1: Comparison of the asymptotic performance measures $\Psi(\text{MANOVA}, \beta, \gamma)$, $\Psi(\text{Marchenko-Pastur}, \beta)$ and $\Psi(\text{Band Limited}, \beta, \gamma)$, as a function of β, for $\gamma = 1/2$ (corresponding to sub-frame aspect ratio β from a redundancy-2 frame, in the families of ETF, random (i.i.d.), and LPF, respectively). Each figure corresponds to a different sub-frame performance measure Ψ: (Left) restricted isometry property (RIP) associated with compressed sensing (lower is better); (Middle) noise amplification associated with MD and CDC (lower is better); (Right) Shannon transform associated with the capacity of NOMA-CDMA (higher is better). The 1/3 line in the left figure is the sharp RIP bound for sparse signal and low rank matrix recovery [21].

that depends on the eigenvalue spectrum of a sub-frame; e.g., noise amplification amounts to the harmonic mean of the spectrum, while capacity (Shannon transform) amounts to the geometric mean of the spectrum. Chapter 4 gives background from frame theory (in particular, earlier results on the spectral properties of sub-frames motivated by compressed sensing), while Chapter 5 gives the relevant background on random matrix theory.

The two highlights of this survey are (i) the *ETF-MANOVA relation*, connecting frame theory with random matrix theory, and (ii) the (still mostly open) possibility of *ETF superiority.* The first highlight is divided between two sections: Chapter 6 describes the empirical results of [74] regarding the universal behavior of sub-frames of ETFs and "near ETFs"; and Chapter 7 develops analytically the convergence to the MANOVA limit distribution based on the moment method (see footnote 4 above) [71], [75], [92]. To support the ETF superiority claim, we examine numerically in Chapter 9 some of the applications listed in Table 1.1; and we prove analytically in Chapter 10 the *erasure Welch bound*, [75], which implies that tight frames have the smallest sub-frame spectral

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variance among all unit-norm frames, and that ETFs have the smallest sub-frame spectral kurtosis among all unit-norm tight frames. In between these two highlights, Chapter 8 proves some sub-frame performance inequalities (in the flavor of the information-theoretic inequalities of [37]), which explain the role of the sub-frame aspect ratio k/m as a design parameter. Finally, Chapter 11 concludes and lists interesting open questions and conjectures that arise in this area.

1.1 Notation

A finite sequence of integers $\{1, \ldots, n\}$ is denoted as $[n]$. Bold letters $\mathbf{f}, \mathbf{x}$ etc. denote column vectors. The $n \times n$ identity matrix is denoted by I_n. Dagger $(\cdot)^\dagger$ denotes transpose or conjugate (Hermitian) transpose, according to the context. The set of all k-subsets of the set $[n]$ is denoted $\binom{[n]}{k}$, or $\{S \subset [n] : |S| = k\}$, or simply $\{S : |S| = k\}$ when the context is clear. $\mathbb{E}$ denotes expectation. Throughout we try to keep the following glossary:

m	vector space dimension
n	frame size
k	sub-frame size
p	selection probability
γ	frame aspect ratio m/n
β	sub-frame aspect ratio k/m
F	$m \times n$ frame matrix
S	k-subset of $[n]$
F_S	$m \times k$ sub-frame matrix
Ψ	performance measure

Full text available at: http://dx.doi.org/10.1561/0100000125

Acknowledgements

Matan Gavish was partially supported by the Israel Science Foundation, grant 1523/16.

Dustin G. Mixon was partially supported by AFOSR FA9550-18-1-0107 and NSF DMS 1829955.

Ram Zamir was partially supported by the Israel Science Foundation, grants 676/15 and 2623/20.

We thank Ofer Zeitouni for suggesting the moment approach to prove the ETF-MANOVA theorem.

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Appendices

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A

Entropy-coded dithered quantization

To asses the rate-distortion performance (2.11) of the analog coding scheme, we shall adopt the additive-noise "test channel" of entropy-coded (subtractive) dithered quantization (ECDQ) [159]. In this model, $Q^{(\text{dither})}(\tilde{\mathbf{x}}) = \tilde{\mathbf{x}} + \mathbf{Z}$, where $\mathbf{Z}$ is an independent uniform or Gaussian noise, and the quantizer entropy is given by the mutual information $H(Q^{(\text{dither})}(\tilde{\mathbf{X}})) = I(\tilde{\mathbf{X}}; \tilde{\mathbf{X}} + \mathbf{Z})$.[1] The equivalent error in the important samples, $\mathbf{E}_s = \hat{\mathbf{X}}_s - \mathbf{X}_s$ is thus, by (2.8), given by $\mathbf{E}_s = F_S^{\dagger} \cdot (\tilde{\mathbf{X}} + \mathbf{Z}) - \mathbf{X}_s = F_S^{\dagger} \cdot \mathbf{Z}$, implying that the mean-squared distortion per important sample (2.2) is

$$D = \frac{1}{k} \mathbb{E} \left\{ \|\mathbf{E}_s\|^2 \right\} = \frac{\sigma_z^2}{k} \cdot \text{trace}\{F_S^{\dagger} F_S\} = \sigma_z^2, \tag{A.1}$$

where σ_z^2 is the quantizer mean-squared error, and the last equation follows since F_S has k unit-norm columns.[2] Since the decoder is blind

[1] To be more precise, the quantizer operation is given by $Q(\tilde{\mathbf{x}} + \text{dither}) - \text{dither}$, where dither is uniform over the fundamental quantizer cell. For good high-dimensional lattice quantizers, this uniform dither distribution tends to be white Gaussian. See [161].

[2] Further improvement can be obtained using minimum mean-squared error "Wiener" estimation, [72], but this is negligible when $\sigma_x^2 \gg D$.

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to the location of important samples (that affect the correlation of the vector $\tilde{\mathbf{X}}$), the coding rate is equal to that of a *white* Gaussian vector $\tilde{\mathbf{W}}$ with the same power as $\tilde{\mathbf{X}}$, [84]. Assuming the LS estimator (2.9), we have $E\|\tilde{\mathbf{X}}\|^2 = \sigma_x^2 \cdot \text{trace}\{(F_S^\dagger F_S)^{-1}\}$, so the ECDQ mutual information formula (with a Gaussian dither) becomes:

$$R_{\text{analog}} = \frac{1}{n} I(\tilde{\mathbf{W}}; \tilde{\mathbf{W}} + \mathbf{Z}) \tag{A.2}$$

$$= \frac{m}{2n} \cdot \log\left(1 + \frac{\frac{1}{m}\mathbb{E}\|\tilde{\mathbf{X}}\|^2}{\sigma_z^2}\right) \tag{A.3}$$

$$= \beta \cdot \frac{p}{2} \cdot \log\left(1 + \frac{\sigma_x^2}{D} \cdot \frac{\text{trace}\{(F_S^\dagger F_S)^{-1}\}}{m}\right) \tag{A.4}$$

where $\beta \triangleq k/m$, and we used (2.2) with $k = pn$.

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B

Information theoretic proofs for sub-frame inequalities

In proving Theorems 8.1 and 8.2 we use the following two lemmas.

Lemma B.1. For any $m \times n$ frame F, and $k < n$,

$$\frac{1}{\binom{n}{k}} \sum_{S:|S|=k} F_S \cdot F_S^\dagger = \frac{k}{n} \cdot F \cdot F^\dagger, \tag{B.1}$$

where the average in the left hand side is over all k-subsets of $\{1, \ldots, n\}$ as in (3.8)-(3.2).

Remark: For Bernoulli(p) selection, i.e., when each index in $\{1, \ldots, n\}$ belongs to S with probability p independently of the other indices, the lemma becomes $\mathbb{E}_S\{F_S \cdot F_S^\dagger\} = p \cdot F \cdot F^\dagger$, where $\mathbb{E}_S\{\cdot\}$ denotes expectation with respect to the selection of S.

Proof: The case $k = 1$ of (B.1) is the standard expansion $\sum_{i=1}^n \mathbf{f}_i \cdot \mathbf{f}_i^\dagger = F \cdot F^\dagger$ (multiplied by $1/n$), where the $\mathbf{f}_i$'s are the columns of F. For a general k, let P denote a random diagonal matrix, whose $\{0, 1\}$ diagonal elements correspond to the selection of the subset S. That is, $F_S \cdot F_S^\dagger = F \cdot P \cdot F^\dagger$, where the diagonal of P is uniform over all n choose k binary vectors with k ones and $n - k$ zeroes. Thus

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$\mathbb{E}_S\{F_S \cdot F_S^\dagger\} = \mathbb{E}_S\{F \cdot P \cdot F^\dagger\} = F \cdot \mathbb{E}_S\{P\} \cdot F^\dagger = F \cdot (k/n \cdot I_n) \cdot F^\dagger$, which is the right-hand side of (B.1). □

Lemma B.2. For $k_1 < k_2 < n$, if S_2 is uniformly drawn from all k_2 subsets of $\{1,\ldots,n\}$, and S_1 is uniformly drawn from all k_1 subsets of S_2, then S_1 is uniform on all k_1 subsets of $[n]$.

Proof: Since the distribution of S_1 is invariant under the action of the symmetric group of $[n]$, it is necessarily uniform. □

Proof of Theorem 8.1: We first prove the inequality with respect to the "edge": $L(F,k) \le L(F,n)$, and then extend to any $k_1 < k_2$. The Ky Fan inequality [34], says that if K_1 and K_2 are $m \times m$ PSD matrices, then

$$\det[a \cdot K_1 + (1-a) \cdot K_2] \ge \det[K_1]^a \cdot \det[K_2]^{1-a} \tag{B.2}$$

for any $0 < a < 1$, which by taking logarithm implies that $\log\det[K]$ is concave [34]. Thus, starting from (3.8),

$$\begin{aligned} L_{\text{Shannon}}(F,k) &= \mathbb{E}_S\{\log[\sqrt[m]{\det(F_S \cdot F_S^\dagger)}/(k/m)]\} && \text{(B.3)}\\ &\le \log[\sqrt[m]{\det(\mathbb{E}_S\{F_S \cdot F_S^\dagger\})}/(k/m)] && \text{(B.4)}\\ &= \log[\sqrt[m]{\det(k/n \cdot F \cdot F^\dagger)}/(k/m)] && \text{(B.5)}\\ &= \log[\sqrt[m]{\det(F \cdot F^\dagger)}/(n/m)] && \text{(B.6)}\\ &= L_{\text{Shannon}}(F,n) && \text{(B.7)} \end{aligned}$$

where the second line is by Jensen's inequality and the log concavity of the determinant (B.2), and the third line is by the identity in Lemma B.1. Turning to the general case, for $k_1 < k_2 < n$, let $S_1 \subset S_2$ denote a k_1-subset of a k_2-subset S_2 of $\{1,\ldots,n\}$. By the definition of the average

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subset Shannon transform of a frame (3.8), we have

$$L_{\text{Shannon}}(F, k_1) = \mathbb{E}_{S_1}\{\Psi_{\text{Shannon}}(F_{S_1})\} \tag{B.8}$$
$$= \mathbb{E}_{S_2}\{E_{S_1|S_2}\{\Psi_{\text{Shannon}}(F_{S_1})\} \tag{B.9}$$
$$= \mathbb{E}_{S_2}\{L_{\text{Shannon}}(F_{S_2}, k_1)\} \tag{B.10}$$
$$\leq \mathbb{E}_{S_2}\{L_{\text{Shannon}}(F_{S_2}, k_2)\} \tag{B.11}$$
$$= \mathbb{E}_{S_2}\{\Psi_{\text{Shannon}}(F_{S_2})\} \tag{B.12}$$
$$= L_{\text{Shannon}}(F, k_2) \tag{B.13}$$

where $\mathbb{E}_{S_1|S_2}\{\cdot\}$ denotes expectation over a uniform distribution on all k_1-subsets of a given S_2, and $\mathbb{E}_{S_2}\{\cdot\}$ denotes expectation over a uniform distribution on all k_2-subsets of $\{1,\ldots,n\}$. The second line follows from Lemma B.2 by iterated expectation; the third line follows by viewing F_{S_2} as the full frame in the inner expectation; the inequality follows from the first part of the proof (B.3)-(B.7) setting $k = k_1$ and $n = k_2$; and the last two lines are again by the definition (3.8). □

Proof of Theorem 8.2: For an $n \times n$ covariance matrix K (i.e., K is a non-negative matrix), and a k-subset $S \subset \{1,\ldots,n\}$, let K_S denote the corresponding $k \times k$ sub-matrix of K. Define the *subset trace-inverse* of the covariance K as the average of trace $\left(K_S^{-1}\right)/k$ over all k-subsets:

$$M_k^{(n)} = \frac{1}{\binom{n}{k}} \sum_{S:|S|=k} \frac{1}{k}\text{trace}\left(K_S^{-1}\right), \tag{B.14}$$

for $k = 1,\ldots,n$. For example, the two extremes are $M_1^{(n)} = 1/n\sum_{i=1}^n (K_{ii})^{-1}$, and $M_n^{(n)} = 1/n\sum_{i=1}^n (K^{-1})_{ii}$. It is shown in [79] that the sequence of subset trace inverses is monotonically non decreasing, i.e.,

$$M_1^{(n)} \leq M_2^{(n)} \leq \ldots \leq M_n^{(n)} \tag{B.15}$$

with equality if and only if the covariance matrix K is proportional to identity.[1] This inequality can be thought of as the "MSE counterpart" of

[1]If the matrix K is Toeplitz (corresponding to stationary vector), then the

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the subset determinant inequality in [34] and [37]. Using similar tools as in [34], we can show the same inequality also for *subset log trace-inverse*. Namely, letting

$$L_k^{(n)} = \frac{1}{\binom{n}{k}} \sum_{S:|S|=k} \log\left(\frac{1}{k}\text{trace}\left(K_S^{-1}\right)\right), \tag{B.16}$$

we have

$$L_1^{(n)} \leq L_2^{(n)} \leq \ldots \leq L_n^{(n)}. \tag{B.17}$$

Proof: Starting with the right edge, noting that $\binom{n}{n} = 1$ and $\binom{n}{n-1} = n$, we have

$$\begin{aligned}
L_n^{(n)} &= \log\left(\frac{1}{n}\text{trace}\left(K^{-1}\right)\right) \\
&= \log\left(M_n^{(n)}\right) \\
&\geq \log\left(M_{n-1}^{(n)}\right) \\
&= \log\left(\frac{1}{n}\sum_{S:|S|=n-1}\frac{1}{n-1}\text{trace}\left(K_S^{-1}\right)\right) \\
&\geq \frac{1}{n}\sum_{S:|S|=n-1}\log\left(\frac{1}{n-1}\text{trace}\left(K_S^{-1}\right)\right) \\
&= L_{n-1}^{(n)},
\end{aligned} \tag{B.18}$$

where the equality lines are by definition; the first inequality follows from (B.15); and the second inequality follows by Jensen. We then continue similarly to the second part of the proof of Theorem 8.1 to prove that $L_k^{(n)} \leq L_{k+1}^{(n)}$ for any k. Specifically, by Lemma B.2, when computing $L_k^{(n)}$ we first condition on a specific $(k+1)$-subset S_2 and average over all its k-subsets, and then average over all $(k+1)$-subsets

inequality between the two edges, $M_1^{(n)} \leq M_n^{(n)}$, follows from the arithmetic-to-harmonic means inequality applied to the eigenvalues $\lambda_1, \ldots, \lambda_n$ of K, and equality holds if and only if the eigenvalues are constant, i.e., $K = \lambda I$. (This is because by the Toeplitz property $K_{ii} = \sigma^2$ for all i, and therefore $M_1^{(n)} = 1/\sigma^2$; furthermore, $\sigma^2 = \text{trace}(K)/n = (\lambda_1 + \ldots + \lambda_n)/n$, while the eigenvalues of K^{-1} are the reciprocals of the eigenvalues of K, so $M_n^{(n)} = (1/\lambda_1 + \ldots + 1/\lambda_n)/n$.)

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S_2 of $\{1,\ldots,n\}$. By (B.18), the inner average is upper bounded by the log trace-inverse of K_{S_2}, which becomes $L_{k+1}^{(n)}$ after taking the outer average. $\square$

Theorem 8.2 now follows from (B.17) by viewing the $F^\dagger \cdot F$ as the covariance matrix K, so $L_{\text{MSE}}(F,k) = L_k^{(n)}$. (Note that for $k > m$ the matrix $F_S^\dagger F_S$ is singular, so $L_k^{(n)}$ is infinite.)

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C

Variance of $1st$ and $2nd$ moments (proof of Theorem 7.2)

Using the moment-variance formula (7.15),

$$\begin{aligned} V_r &\triangleq \mathrm{Var}\left[\frac{1}{n}\mathrm{trace}\left((FPF^{\dagger})^r\right)\right] \\ &= \mathbb{E}\left[\left(\frac{1}{n}\mathrm{trace}\left((FPF^{\dagger})^r\right)\right)^2\right] - m_r^2 \end{aligned}$$

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we obtain for $r = 1$,

$$\begin{aligned}
V_1^{\mathrm{ETF}} &= \mathbb{E}\left[\left(\frac{1}{n}\mathrm{trace}\left(FPF^\dagger\right)\right)^2\right] - \mathbb{E}\left[\frac{1}{n}\mathrm{trace}\left(FPF^\dagger\right)\right]^2 \\
&= \mathbb{E}\left[\left(\frac{1}{n}\mathrm{trace}\left(FPF^\dagger\right)\right)^2\right] - m_1^2 \\
&= \frac{1}{n^2}\mathbb{E}\left[\left(\sum_{i=1}^{n}\langle \mathbf{f}_i, \mathbf{f}_i\rangle P_i\right)\left(\sum_{j=1}^{n}\langle \mathbf{f}_j, \mathbf{f}_j\rangle P_j\right)\right] - p^2 \\
&= \frac{1}{n^2}\mathbb{E}\left[\left(\sum_{i=1}^{n} P_i\right)\left(\sum_{j=1}^{n} P_j\right)\right] - p^2 \\
&= \frac{1}{n^2}\left[np + n(n-1)p^2\right] - p^2 \\
&= \frac{1}{n}\left[p - p^2\right]. \qquad\qquad (C.1)
\end{aligned}$$

For $r = 2$, we obtain

$$\begin{aligned}
V_2^{\mathrm{ETF}} &= \mathbb{E}\left[\left(\frac{1}{n}\mathrm{trace}\left((FPF^\dagger)^2\right)\right)^2\right] - \mathbb{E}\left[\frac{1}{n}\mathrm{trace}\left((FPF^\dagger)^2\right)\right]^2 \\
&= \mathbb{E}\left[\left(\frac{1}{n}\mathrm{trace}\left((FPF^\dagger)^2\right)\right)^2\right] - m_2^2 \\
&= \frac{1}{n^2}\mathbb{E}\left[\left(\sum_{i,j=1}^{n} |\langle \mathbf{f}_i, \mathbf{f}_j\rangle|^2 P_i P_j\right)\left(\sum_{k,m=1}^{n} |\langle \mathbf{f}_k, \mathbf{f}_m\rangle|^2 P_k P_m\right)\right] - (p + p^2 x)^2. \qquad (C.2)
\end{aligned}$$

If F is an ETF, then according to the Welch bound $|\langle \mathbf{f}_i, \mathbf{f}_j\rangle|^2 = \frac{x}{n-1}$ for $i \neq j$. Similarly to the computation of the moments, we split to cases by number of distinct values of $\{i, j, k, m\}$.

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$$\mathbb{E}\left[\left(\sum_{i,j=1}^{n}|\langle \mathbf{f}_i,\mathbf{f}_j\rangle|^2 P_iP_j\right)\left(\sum_{k,m=1}^{n}|\langle \mathbf{f}_k,\mathbf{f}_m\rangle|^2 P_kP_m\right)\right] = np$$
$$+\left[n(n-1)+4n(n-1)\frac{x}{n-1}+2n(n-1)\frac{x^2}{(n-1)^2}\right]p^2$$
$$+\left[2n(n-1)(n-2)\frac{x}{n-1}+4n(n-1)(n-2)\frac{x^2}{(n-1)^2}\right]p^3$$
$$+n(n-1)(n-2)(n-3)\frac{x^2}{(n-1)^2}p^4. \quad \text{(C.3)}$$

It follows that

$$V_2 = \frac{1}{n}\Big[p+\left(-1+4x+\frac{2x^2}{n-1}\right)p^2+\left(-4x+\frac{4(n-2)x^2}{n-1}\right)p^3$$
$$+\left(\frac{(6-4n)x^2}{n-1}\right)p^4\Big]. \quad \text{(C.4)}$$

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