Bifurcation of Macroeconometric Models and Robustness of Dynamical Inferences

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Contents

1 Bifurcation of Macroeconomic Models **3**
1.1 Introduction . . . 3
1.2 Stability . . . 6
1.3 Types of bifurcations . . . 8

2 Bergstrom–Wymer Continuous Time UK Model **20**
2.1 Introduction . . . 20
2.2 The model . . . 22
2.3 Stability of the equilibrium . . . 30
2.4 Determination of bifurcation boundaries . . . 33
2.5 Stabilization policy . . . 39

3 Leeper and Sims Model **43**
3.1 Introduction . . . 43
3.2 The model . . . 45
3.3 Singularity in Leeper and Sims model . . . 52
3.4 Numerical results . . . 59

4 New Keynesian Model **62**
4.1 Introduction . . . 62
4.2 The model . . . 63
4.3 Determinacy and stability analysis . . . 64

5 New Keynesian Model with Regime Switching **81**
5.1 Introduction 81
5.2 Dynamics with a simple monetary policy rule 83
5.3 New Keynesian model with regime switching 87
5.4 New Keynesian model with a hybrid monetary policy rule . 89

6 Zellner's Marshallian Macroeconomic Model **91**
6.1 Introduction 91
6.2 The model 92
6.3 Stability and bifurcation analysis of equilibrium 96

7 Open-Economy New Keynesian Models **99**
7.1 Introduction 99
7.2 Gali and Monacelli model 101
7.3 Clarida et al. model 112

8 Two Endogenous Growth Models **115**
8.1 Introduction 115
8.2 Uzawa–Lucas endogenous growth model 117
8.3 Jones semi-endogenous growth model 125

9 Conclusion **135**

References **137**

Abstract

In systems theory, it is well known that the parameter spaces of dynamical systems are stratified into bifurcation regions, with each supporting a different dynamical solution regime. Some can be stable, with different characteristics, such as monotonic stability, periodic damped stability, or multiperiodic damped stability, and some can be unstable, with different characteristics, such as periodic, multiperiodic, or chaotic unstable dynamics. But in general the existence of bifurcation boundaries is normal and should be expected from most dynamical systems, whether linear or nonlinear. Bifurcation boundaries in parameter space are not evidence of model defect. While existence of such bifurcation boundaries is well known in economic theory, econometricians using macroeconometric models rarely take bifurcation into consideration, when producing policy simulations from macroeconometrics models. Such models are routinely simulated only at the point estimates of the models' parameters.

Barnett and He [1999] explored bifurcation stratification of Bergstrom and Wymer's [1976] continuous time UK macroeconometric model. Bifurcation boundaries intersected the confidence region of the model's parameter estimates. Since then, Barnett and his coauthors have been conducting similar studies of many other newer macroeconometric models spanning all basic categories of those models. So far, they have not found a single case in which the model's parameter space was not subject to bifurcation stratification. In most cases, the confidence region of the parameter estimates were intersected by some of those bifurcation boundaries. The most fundamental implication of this research is that policy simulations with macroeconometric models should be conducted at multiple settings of the parameters within the confidence region. While this result would be as expected by systems theorists, the result contradicts the normal procedure in macroeconometrics of conducting policy simulations solely at the point estimates of the parameters.

This survey provides an overview of the classes of macroeconometric models for which these experiments have so far been run and emphasizes the implications for lack of robustness of conventional dynamical

inferences from macroeconometric policy simulations. By making this detailed survey of past bifurcation experiments available, we hope to encourage and facilitate further research on this problem with other models and to emphasize the need for simulations at various points within the confidence regions of macroeconometric models, rather than at only point estimates.

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1

Bifurcation of Macroeconomic Models[1]

1.1 Introduction

Bifurcation has long been a topic of interest in dynamical macroeconomic systems. Bifurcation analysis is important in understanding dynamic properties of macroeconomic models as well as in selection of stabilization policies. The goal of this survey is to summarize work by William A. Barnett and his coauthors on bifurcation analyses in macroeconomic models to facility and motivate work by others on further models. In Section 1, we introduce the concept of bifurcation and its role in studies of macroeconomic systems and also discuss several types of bifurcations by providing examples summarized from Barnett and He [2004, 2006a,b]. In Sections 2–8, we discuss bifurcation analysis and approaches with models from Barnett's other papers on this subject.

To explain what bifurcation is, Barnett and He [2004, 2006a,b] begin with the general form of many existing macroeconomic models:

$$\mathbf{D}\mathbf{x} = \mathbf{f}(\mathbf{x},\boldsymbol{\theta}), \tag{1.1}$$

[1] This section is summarized from Barnett and Binner [2004], Barnett and He [2006a,b].

where $\mathbf{D}$ is the vector-valued differentiation operator, $\mathbf{x}$ is the state vector, $\boldsymbol{\theta}$ is the parameter vector, and $\mathbf{f}$ is the vector of functions governing the dynamics of the system, with each component assumed to be smooth in a local region of interest.

In system (1.1), the focus of interest lies in the settings of the parameter vector, $\boldsymbol{\theta}$. Assume $\boldsymbol{\theta}$ takes values within a theoretically feasible set $\boldsymbol{\Theta}$. The value of $\boldsymbol{\theta}$ can affect the dynamics of the system substantially through a small change, and we say a bifurcation occurs in the system, if such a small change in parameters fundamentally alters the nature of the dynamics of the system. In particular, bifurcation refers to a change in qualitative features instead of quantitative features of the solution dynamics. A change in quantitative features of dynamical solutions may refer to a change in such properties as the period or amplitude of cycles, while a change in qualitative features may refer to such changes as changes from one type of stability or instability to another type of stability or instability.

A point within the parameter space at which a change in qualitative features of the dynamical solution path occurs defines a point on a bifurcation boundary. At the bifurcation point, the structure of the dynamic system may change fundamentally. Different dynamical solution properties can occur when parameters are close to but on different sides of a bifurcation boundary. A parameter set can be stratified by bifurcation boundaries into several subsets with different types of dynamics within each subset.

There are several types of bifurcation boundaries, such as Hopf, pitchfork, saddle-node, transcritical, and singularity bifurcation. Each type of bifurcation produces a different type of qualitative dynamic change. We illustrate these different types of bifurcation by providing examples in Section 1.3. Bifurcation boundaries have been discovered in many macroeconomic systems. For example, Hopf bifurcations have been found in growth models [e.g., Benhabib and Nishimura, 1979, Boldrin and Woodford, 1990, Dockner and Feichtinger, 1991, Nishimura and Takahashi, 1992], and in overlapping generations models. Pitchfork bifurcations have been found in the tatonnement process [e.g., Bala, 1997, Scarf, 1960]. Transcritical bifurcations have been found in Bergstrom and Wymer's [1976] UK model [Barnett and He, 1999] and singularity bifurcation in Leeper and Sims' Euler-equation model [Barnett and Duzhak, 2008].

One reason we are concerned about bifurcation phenomena in macroeconomic models is because changes in parameters could affect dynamic behaviors of the models and consequently the outcomes of imposition of policy rules. For example, Bergstrom and Wymer's [1976] UK model operates close to bifurcation boundaries between stable and unstable regions of the parameter space. In this case, if a bifurcation boundary intersects the confidence region of the parameter estimates, different qualitative properties of solution can exist within this confidence region. As a result, robustness of inferences about dynamics can be damaged, especially if inferences about dynamics are based on model simulations with the parameters set only at their point estimates. When confidence regions are stratified by bifurcation boundaries, dynamical inferences need to be based on simulations at points within each of the stratified subsets of the confidence region.

Knowledge of bifurcation boundaries is directly useful in policy selection. If the system is unstable, a successful policy would bifurcate the system from the unstable to stable region. In that sense, stabilization policy can be viewed as bifurcation selection. As illustrated in Section 2, Barnett and He [2002] have shown that successful bifurcation policy selection can be difficult to design.

Barnett's work has found bifurcation phenomena in every macroeconomic model that he and his coauthors have so far explored. Barnett and He [1999, 2002] examined the dynamics of Bergstrom-Wymer's continuous-time dynamic macroeconomic model of the UK economy and found both transcritical and Hopf bifurcation boundaries. Barnett and He [2008] estimated and displayed singularity bifurcation boundaries for the Leeper and Sims's [1994] Euler equations model. Barnett and Duzhak [2010] found Hopf and period doubling bifurcations in a New Keynesian model. Banerjee et al. [2011] examined the possibility of cyclical behavior in the Marshallian Macroeconomic Model. Barnett and Eryilmaz [2013, 2014], investigated bifurcation in open economy models. Barnett and Ghosh [2013] investigated the existence of bifurcations in endogenous growth models.

This survey is organized in the chronological order of Barnett's work on bifurcation of macroeconomic models, from early models to many of the most recent models.

1.2 Stability

There are two possible approaches to analyze bifurcation phenomena: global and local. Methods in Barnett's current papers have used local analysis, which is analysis of the linearized dynamic system in a neighborhood of the steady state. In his papers, (1.1) is linearized in the form

$$\mathbf{D}\mathbf{x} = \mathbf{A}(\theta)\mathbf{x} + \mathbf{F}(\mathbf{x}, \theta), \tag{1.2}$$

where $\mathbf{A}(\theta)$ is the Jacobian matrix of $\mathbf{f(x,\theta)}$, and $\mathbf{F(x,\theta)} = \mathbf{f(x,\theta) - A(\theta)x = o(x,\theta)}$ is the vector of higher order term. Define $\mathbf{x}^*$ to be the system's steady state equilibrium, such that $\mathbf{f(x^*,\theta) = 0}$, and redefine the variables such that the steady state is the point $\mathbf{x^* = 0}$ by replacing $\mathbf{x}$ with $\mathbf{x - x^*.}$

The local stability of (1.1), for small perturbation away from the equilibrium, can be studied through the eigenvalues of $\mathbf{A(\theta)}$, which is a matrix-valued function of the parameters θ. It is important to know at what parameter values, θ, the system (1.1) is unstable. But it is also important to know the nature of the instability, such as periodic, multiperiodic, or chaotic, and the nature of the stability, such as monotonically convergent, damped single-periodic convergent, or damped multiperiodic convergent. For global analysis, which can be far more complicated than local analysis, higher order terms must be considered, since the perturbations away from the equilibrium can be large. Analysis of $\mathbf{A(\theta)}$ alone may not be adequate. More research on global analysis of macroeconomic models is needed.

To analyze the local stability properties of the system, we need to locate the bifurcation boundaries. The boundaries must satisfy

$$det(\mathbf{A}(\theta)) = \mathbf{0}. \tag{1.3}$$

According to Barnett and Binner [2004], if all eigenvalues of $\mathbf{A(\theta)}$ have strictly negative real parts, then (1.1) is locally asymptotically stable in the neighborhood of $\mathbf{x} = \mathbf{0}$. If at least one of the eigenvalues of $\mathbf{A}(\boldsymbol{\theta})$ has positive real part, then (1.1) is locally asymptotically unstable in the neighborhood of $\mathbf{x} = \mathbf{0}$.

The bifurcation boundaries can be difficult to locate. In Barnett and He [1999, 2002], various methods are applied to locate the bifurcation boundaries characterized by (1.3). Equation (1.3) usually cannot be solved in closed form, when $\boldsymbol{\theta}$ is multi-dimensional. As a result, numerical methods are extensively used for solving (1.3).

Before proceeding to the next section, we introduce the definition of hyperbolic for flows and maps, respectively. According to Hale and Kocak [1991], the following definitions apply.

Definition 1.1. An equilibrium point $\mathbf{x}^*$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is said to be *hyperbolic*, if all the eigenvalues of the Jacobian matrix $D\mathbf{f}(\mathbf{x}^*)$ have nonzero real parts.

Definition 1.2. A fixed point $\mathbf{x}^*$ of $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x})$ is said to be *hyperbolic*, if the linear C^1 map $\mathbf{x} \mapsto D\mathbf{f}(\mathbf{x}^*)\mathbf{x}$ is *hyperbolic*; that is, if the Jacobian matrix $D\mathbf{f}(\mathbf{x}^*)$ at $\mathbf{x}^*$ has no eigenvalues with modulus one.

Definition 1.2 refers to discrete-time dynamical systems. Since bifurcations can only occur in a local neighborhood of nonhyperbolic equilibria, we are more interested in the behavior at nonhyperbolic equilibria.

For a discrete-time dynamical system, consider a generic smooth one-parameter family of maps $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \boldsymbol{\alpha}) = \mathbf{f}_{(\boldsymbol{\alpha})}(\mathbf{x}), \mathbf{x} \in R^n$, $\alpha \in R$. Since local bifurcation happens only at nonhyperbolic fixed points, there are three critical cases to consider:

(a) The fixed point $\mathbf{x}^*$ has eigenvalue 1.

(b) The fixed point $\mathbf{x}^*$ has eigenvalue -1.

(c) The fixed point $\mathbf{x}^*$ has a pair of complex-conjugate eigenvalues $e^{\pm i\theta_0}$ with $0 < \theta_0 < \pi$.

The codimension 1 bifurcation associated with case (a) is called a *fold* (*saddle node*) bifurcation. The codimension 1 bifurcation associated with case (b) is called a *flip* (*period doubling*) bifurcation, while the codimension 1 bifurcation associated with case (c) is called a *Neimark-Sacker* bifurcation. *Neimark-Sacker* bifurcation is the equivalent of *Hopf* bifurcation for maps.

In the following section, we are going to introduce three important one-dimensional equilibrium bifurcations described locally by ordinary differential equations. They are transcritical, pitchfork, and saddle-node bifurcations.

1.3 Types of bifurcations

1.3.1 Transcritical bifurcations

For a one-dimensional system,

$$Dx = G(x, \theta),$$

the transversality conditions for a transcritical bifurcation at $(x, \theta) = (0, 0)$ are

$$G(0,0) = G_x(0,0) = G_\theta(0,0) = 0, \; G_{xx}(0,0) \neq 0, \quad \text{and}$$
$$G_{\theta x}^2 - G_{xx}G_{\theta\theta}(0,0) > 0. \tag{1.4}$$

An example of such a form is

$$Dx = \theta x - x^2. \tag{1.5}$$

The steady state equilibria of the system are at $x^* = 0$ and $x^* = \theta$. It follows that system (1.5) is stable around the equilibrium $x^* = 0$ for $\theta < 0$, and unstable for $\theta > 0$. System (1.5) is stable around the equilibrium $x^* = \theta$ for $\theta > 0$, and unstable for $\theta < 0$. The nature of the dynamics changes as the system bifurcates at the origin. This transcritical bifurcation arises in systems in which there is a simple solution branch, corresponding here to $x^* = 0$.

Transcritical bifurcations have been found in high-dimensional continuous-time macroeconomic systems, but in high dimensional cases, transversality conditions have to be verified on a manifold. Details are provided in Guckenheimer and Holmes [1983].

1.3.2 Pitchfork bifurcations

For a one-dimensional system,

$$Dx = f(x, \theta).$$

Suppose that there exists an equilibrium x^* and a parameter value θ^* such that (x^*, θ^*) satisfies the following conditions:

(a) $$\left.\frac{\partial f(x, \theta^*)}{\partial x}\right|_{x=x^*} = 0,$$

(b) $$\left.\frac{\partial^3 f(x, \theta^*)}{\partial x^3}\right|_{x=x^*} \neq 0,$$

(c) $$\left.\frac{\partial^2 f(x, \theta)}{\partial x \partial \theta}\right|_{x=x^*, \theta=\theta^*} \neq 0,$$

then (x^*, θ^*) is a pitchfork bifurcation point.

An example of such form is

$$Dx = \theta x - x^3.$$

The steady state equilibria of the system are at $x^* = 0$ and $x^* = \pm\sqrt{\theta}$. It follows that the system is stable when $\theta < 0$ at the equilibrium $x^* = 0$, and unstable at this point when $\theta > 0$. The two other equilibria $x^* = \pm\sqrt{\theta}$ are stable for $\theta > 0$. The equilibrium $x^* = 0$ loses stability, and two new stable equilibria appear. This pitchfork bifurcation, in which a stable solution branch bifurcates into two new equilibria as θ increases, is called a supercritical bifurcation.

Bala [1997] shows how pitchfork bifurcation can occur in the tatonnement process.

1.3.3 Saddle-Node bifurcations

For a one-dimensional system,

$$Dx = f(x, \theta).$$

A saddle-node point (x^*, θ^*) satisfies the equilibrium condition $f(x^*, \theta^*) = 0$ and the Jacobian condition

$$\left.\frac{\partial f(x, \theta^*)}{\partial x}\right|_{x=x^*} = 0,$$

as well as the transversality conditions, as follows:

(a) $$\frac{\partial f(x,\theta)}{\partial \theta}|_{x=x^*,\theta=\theta^*} \neq 0,$$

(b) $$\frac{\partial^2 f(x,\theta)}{\partial x^2}|_{x=x^*,\theta=\theta^*} \neq 0.$$

Sotomayor [1973] shows that transversality conditions for high-dimensional systems can also be formulated.

A simple system with a saddle-node bifurcation is

$$Dx = \theta - x^2.$$

The equilibria are at $x^* = \pm\sqrt{\theta}$, which requires θ to be nonnegative. Therefore, there exist no equilibria for $\theta < 0$, and there exist two equilibria at $x^* = \pm\sqrt{\theta}$, when $\theta > 0$. It follows that when $\theta > 0$, the system is stable at $x^* = \sqrt{\theta}$ and unstable at $x^* = -\sqrt{\theta}$. In this example, bifurcation occurs at the origin as θ increases through zero, which is called the (supercritical) saddle node.

1.3.4 Hopf bifurcations

Hopf bifurcation is the most studied type of bifurcation in economics. For continuous time systems, Hopf bifurcation occurs at the equilibrium points at which the system has a Jacobian matrix with a pair of purely imaginary eigenvalues and no other eigenvalues which have zero real parts. For discrete time system, the following theorem applies in the special case of $n = 2$. The Hopf bifurcation theorem in Gandolfo (2010, ch. 24, p. 497) is widely applied to find the existence of Hopf bifurcation.

Theorem 1.1 Existence of Hopf bifurcation in two dimensions. Consider the two-dimensional nonlinear difference system with one parameter

$$\mathbf{y}_{t+1} = \boldsymbol{\varphi}(\mathbf{y}_t, \alpha),$$

and suppose that for each α in the relevant interval there exists a smooth family of equilibrium points, $\mathbf{y}_e = \mathbf{y}_e(\alpha)$, at which the eigenvalues are complex conjugates, $\lambda_{1,2} = \theta(\alpha) + i\omega(\alpha)$. If there is a critical value α_0 of the parameter such that

(a) the eigenvalues' modulus becomes unity at α_0 but the eigenvalues are not roots of unity (from the first up to the fourth), namely

$$|\lambda_{1,2}(\alpha_0)| = \sqrt{\theta^2 + \omega^2} = 1, \quad \lambda_{1,2}^j(\alpha_0) \neq 1 \quad \text{for } j = 1, 2, 3, 4,$$

and

(b) $\frac{d|\lambda_{1,2}(\alpha)|}{d\alpha}|_{\alpha=\alpha_0} \neq 0$,

then there is an invariant closed curve bifurcating from α_0.

This theorem only applies with a 2×2 Jacobian. The earliest theoretical works on Hopf bifurcation include Poincaré [1892] and Andronov [1929], both of which were concerned with two-dimensional vector fields. A general theorem on the existence of Hopf bifurcation, which is valid in n dimensions, was proved by Hopf [1942].

A simple example in the two-dimensional system is

$$Dx = -y + x(\theta - (x^2 + y^2)),$$
$$Dy = x + y(\theta - (x^2 + y^2)).$$

One equilibrium is $x^* = y^* = 0$ with stability occurring for $\theta < 0$ and the instability occurring for $\theta > 0$. That equilibrium has a pair of conjugate eigenvalues $\theta + i$ and $\theta - i$. The eigenvalues become purely imaginary, when $\theta = 0$.

Barnett and Binner [2004] show the following method to find Hopf bifurcation. They let $p(s) = \det(s\mathbf{I} - \mathbf{A})$ be the characteristic polynomial of $\mathbf{A}$ and write it as

$$p(s) = c_0 + c_1 s + c_2 s^2 + c_3 s^3 + \cdots + c_{n-1} s^{n-1} + s^n.$$

They construct the following $(n-1) \times (n-1)$ matrix

$$\mathbf{S} = \begin{bmatrix} c_0 & c_2 & \ldots & c_{n-2} & 1 & 0 & 0 & \ldots & 0 \\ 0 & c_0 & c_2 & \ldots. & c_{n-2} & 1 & 0 & \ldots & 0 \\ & & & & \ldots & & & & \\ 0 & 0 & \ldots & 0 & c_0 & c_2 & c_4 & \ldots & 1 \\ c_1 & c_3 & \ldots & c_{n-1} & 0 & 0 & 0 & \ldots & 0 \\ 0 & c_1 & c_3 & \ldots & c_{n-1} & 0 & 0 & \ldots & 0 \\ & & & & \ldots & & & & \\ & & & & \ldots & & & & \\ 0 & 0 & \ldots & 0 & c_1 & c_3 & \ldots & \ldots & c_{n-1} \end{bmatrix}$$

Let $\mathbf{S}_0$ be obtained by deleting rows 1 and $n/2$ and columns 1 and 2, and let $\mathbf{S}_1$ be obtained by deleting rows 1 and $n/2$ and columns 1 and 3. The matrix $\mathbf{A}(\theta)$ has one pair of purely imaginary eigenvalues [Guckenheimer et al., 1997], if

$$\det(\mathbf{S}) = 0, \quad \det(\mathbf{S}_0)\det(\mathbf{S}_1) > 0. \tag{1.6}$$

If $\det(\mathbf{S}) = 0$ and $\det(\mathbf{S}_0)\det(\mathbf{S}_1) = 0$, then $\mathbf{A}(\theta)$ may have more than one pair of purely imaginary eigenvalues. The following condition can be used to find candidates for bifurcation boundaries:

$$\det(\mathbf{S}) = 0, \quad \det(\mathbf{S}_0)\det(\mathbf{S}_1) \geq 0. \tag{1.7}$$

Since solving (1.7) analytically is difficult, Barnett and He [1999] apply the following numerical procedure to find bifurcation boundaries. Without loss of generality, they initially consider only two parameters θ_1 and θ_2.

Procedure (P1)

(1) For any fixed θ_1, treat θ_2 as a function of θ_1, and find the value of θ_2 satisfying the condition $h(\theta_2) = \det(\mathbf{A}(\theta)) = 0$. First find the number of zeros of $h(\theta_2)$. Starting with approximations of zeros, use the following gradient algorithm to find all zeros of $h(\theta_2)$:

$$\theta_2(n+1) = \theta_2(n) - a_n h(\theta_2)|_{\theta_2 = \theta_2(n)} \tag{1.8}$$

where $\{a_n, n = 0, 1, 2, \ldots\}$ is a sequence of positive step sizes.

(2) Repeat the same procedure to find all θ_2 satisfying (1.7).

(3) Plot all the pairs (θ_1, θ_2).

(4) Check all parts of the plot to find the segments representing the bifurcation boundaries. Then parts of the curve found in Step (1) are boundaries of saddle-node bifurcations. Parts of the curve found in Step (2) are boundaries of Hopf bifurcations, if the required transversality conditions are satisfied.

Pioneers in studies of Hopf bifurcations in economics include Torre [1977] and Benhabib and Nishimura [1979]. Torre found the appearance of a limit cycle associated with a Hopf bifurcation boundary in Keynesian systems. Benhabib and Nishimura showed that a closed invariant curve might emerge as the result of optimization in a multi-sector neoclassical optimal growth model. These studies illustrate the existence of a Hopf bifurcation boundary in an economic model results in a solution following closed curves around the stationary state. The solution paths may be stable or unstable, depending upon the side of the bifurcation boundary on which the parameter values lie. More recent studies finding Hopf bifurcation in econometric models include Barnett and He [1999, 2002, 2008], who found bifurcation boundaries of the Bergstrom–Wymer continuous-time UK model and the Leeper and Sims Euler-equations model.

1.3.5 Singularity-induced bifurcations

This section is devoted to a dramatic kind of bifurcation found by Barnett and He [2008] in the Leeper and Sims [1977] model — singularity-induced bifurcation.

Some macroeconomic models, such as the dynamic Leontief model [Luenberger and Arbel, 1977] and the Leeper and Sims [1994] model, have the form

$$\mathbf{B}\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t). \tag{1.9}$$

Here $\mathbf{x}(t)$ is the state vector, $\mathbf{f}(t)$ is the vector of driving variables, t is time, and $\mathbf{B}$ and $\mathbf{A}$ are constant matrices of appropriate dimensions.

If $\mathbf{f}(t) = \mathbf{0}$, the system (1.9) is in the class of autonomous systems. Barnett and He [2006a,b] illustrate only the autonomous cases of (1.9).

If $\mathbf{B}$ is invertible, then we can invert $\mathbf{B}$ to acquire

$$\mathbf{x}(t+1) = \mathbf{B}^{-1}\mathbf{A}\mathbf{x}(t) + \mathbf{B}^{-1}\mathbf{f}(t),$$

so that

$$\begin{aligned}\mathbf{x}(t+1) - \mathbf{x}(t) &= \mathbf{B}^{-1}\mathbf{A}\mathbf{x}(t) - \mathbf{x}(t) + \mathbf{B}^{-1}\mathbf{f}(t)\\ &= (\mathbf{B}^{-1}\mathbf{A} - \mathbf{I})\mathbf{x}(t) + \mathbf{B}^{-1}\mathbf{f}(t),\end{aligned}$$

which is in the form of (1.1).

The case in which the matrix $\mathbf{B}$ is singular is of particular interest. Barnett and He [2006a,b] rewrite (1.9) by generalizing the model to permit nonlinearity as follows:

$$\mathbf{B}(\mathbf{x}(t), \boldsymbol{\theta})\mathbf{D}\mathbf{x} = \mathbf{F}(\mathbf{x}(t), \mathbf{f}(t), \boldsymbol{\theta}). \tag{1.10}$$

Here $\mathbf{f}(t)$ is the vector of driving variables, and t is time. Barnett and He [2006a,b] consider the autonomous cases in which $\mathbf{f}(t) = \mathbf{0}$.

Singularity-induced bifurcation occurs, when the rank of $\mathbf{B}(\mathbf{x}, \boldsymbol{\theta})$ changes, as from an invertible matrix to a singular one. Therefore, the matrix must depend on $\boldsymbol{\theta}$ for such changes to occur. If the rank of $\mathbf{B}(\mathbf{x}, \boldsymbol{\theta})$ does not change according to the change of $\boldsymbol{\theta}$, then singularity of $\mathbf{B}(\mathbf{x}, \boldsymbol{\theta})$ is not sufficient for (1.10) to be able to produce singularity bifurcation.

Barnett and He [2006a,b] consider the two-dimensional state-space case and perform an appropriate coordinate transformation allowing (1.10) to become the following equivalent form:

$$\begin{aligned}\mathrm{B}_1(x_1, x_2, \boldsymbol{\theta})\mathrm{D}\mathrm{x}_1 &= F_1(x_1, x_2, \boldsymbol{\theta}),\\ 0 &= \mathrm{F}_2(x_1, x_2, \boldsymbol{\theta}).\end{aligned}$$

They provide four examples to demonstrate the complexity of bifurcation behaviors that can be produced from system (1.10). The first two examples do not produce singularity bifurcations, since $\mathbf{B}$ does not depend on $\boldsymbol{\theta}$. In the second two examples, Barnett and Duzhak [2008] find singularity bifurcation, since $\mathbf{B}$ does depend on $\boldsymbol{\theta}$.

Example 1.1. Consider the following system modified from system (1.5), which has been shown to produce transcritical bifurcation:

$$Dx = \theta x - x^2, \tag{1.11}$$

$$0 = x - y^2. \tag{1.12}$$

Comparing with the general form of (1.10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which is singular but does not depend upon the value of θ.

The equilibria are $(x^*, y^*) = (0,0)$ and $(\theta, \pm\sqrt{\theta})$ Near the equilibrium $(x^*, y^*) = (0,0)$, the system ((1.11),(1.12)) is stable for $\theta < 0$ and unstable for $\theta > 0$. The equilibria $(x^*, y^*) = (\theta, \pm\sqrt{\theta})$ are undefined, when $\theta < 0$, and stable when $\theta > 0$. The bifurcation point is $(x, y, \theta) = (0, 0, 0)$. Notice before and after bifurcation, the number of differential equations and the number of algebraic equations remain unchanged. This implies that the bifurcation point does not produce singularity bifurcation, since $\mathbf{B}$ does not depend upon θ.

Example 1.2. Consider the following system modified from system (1.7), which can produce saddle-node bifurcation:

$$Dx = \theta - x^2, \tag{1.13}$$

$$0 = x - y^2. \tag{1.14}$$

Comparing with the general form of (1.10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which is singular but does not depend upon the value of θ.

The equilibria are at $(x^*, y^*) = (\sqrt{\theta}, \pm\sqrt[4]{\theta})$, defined only for $\theta \geq 0$. The system ((1.13),(1.14)) is stable around both of the equilibria $(x^*, y^*) = (\sqrt{\theta}, \pm\sqrt[4]{\theta})$ and $(x^*, y^*) = (\sqrt{\theta}, \pm\sqrt[4]{\theta})$. The bifurcation point is $(x^*, y^*, \theta) = (0, 0, 0)$. The three-dimensional bifurcation diagram in

Barnett and He [2006a,b] shows that there is no discontinuity or change in dimension at the origin at the origin. The bifurcation point does not produce singularity bifurcation, since the dimension of the state space dynamics remains unchanged on either side of the origin.

Example 1.3. Consider the following system:

$$Dx = ax - x^2, \quad \text{with} \quad a > 0, \tag{1.15}$$

$$\theta Dy = x - y^2. \tag{1.16}$$

Comparing with the general form of (1.10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix},$$

which does depend upon the parameter θ.

When $\theta = 0$, the system has one differential equation (1.15) and one algebraic equation (1.16). If $\theta \neq 0$, the system has two differential equations (1.15) and (1.16) with no algebraic equations for nonzero θ.

The equilibria are $(x^*, y^*) = (0, 0)$ and $(a, \pm\sqrt{a})$. For any value of θ, the system ((1.15),(1.16)) is unstable around the equilibrium at $(x^*, y^*) = (0, 0)$. The equilibrium $(x^*, y^*) = (a\sqrt{a})$ is unstable for $\theta < 0$ and stable for $\theta > 0$. The equilibrium $(x^*, y^*) = (a, -\sqrt{a})$ is unstable for $\theta > 0$ and stable for $\theta < 0$.

Without loss of generality, Barnett and He [2006a,b] normalize a to be 1. When $\theta = 0$, the system's behavior degenerates into movement along the one-dimensional curve $x - y^2 = 0$ When $\theta \neq 0$, the dynamics of the system move throughout the two-dimensional state space. The singularity bifurcation caused by the transition from nonzero θ to zero results in the drop in the dimension.

Barnett and He [2006a,b] observe that even if singularity bifurcation does not cause a change of the system between stability and instability, dynamical properties produced by singularity bifurcation can change. For example, if θ changes from positive to zero, when (x, y) is at the equilibrium $(1, 1)$, the system will remain stable; if θ changes from positive to zero, when (x, y) is at the equilibrium $(0, 0)$, the system will remain unstable; if θ changes from positive to zero, when (x, y)

is at the equilibrium $(1,-1)$, the system will change from unstable to stable. But in all of these cases, the nature of the disequilibrium dynamics changes dramatically, even if there is no transition between stability and instability.

Example 1.4. Consider the following system:

$$Dx = ax - x^2, \quad \text{with} \quad a > 0, \tag{1.17}$$

$$\theta Dy = x - y. \tag{1.18}$$

Comparing with the general form of (1.10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix}.$$

The equilibria are $(x^*, y^*) = (0,0)$ and (a,a). The system is unstable around the equilibrium $(x^*, y^*) = (0,0)$ for any value of θ. The equilibrium $(x^*, y^*) = (a,a)$ is unstable for $\theta < 0$ and stable for $\theta \geq 0$. When $\theta < 0$, the system is unstable everywhere. When $\theta = 0$, equation (1.18) becomes the algebraic constraint $y = x$, which is a one-dimensional ray through the origin. However, when $\theta \neq 0$, the system moves into the two-dimensional space. Even though the dimension can drop from singular bifurcation, there could be no change between stability and instability. For example, $(0,0)$ remains unstable and $(1,1)$ remains stable, when $\theta \neq 0$ and $\theta = 0$.

Barnett and He [2006a,b] also observe that the nature of the dynamics with θ small and positive is very different from the dynamics with θ small and negative. In particular, the equilibrium at $(x^*, y^*) = (1,1)$ is stable in the former case and unstable in the latter case. Hence there is little robustness of dynamical inference to small changes of θ close to the bifurcation boundary. Barnett and Binner [2004, Part 4] further investigate the subject of robustness of inferences in dynamic models.

Example 1.5. Consider the following system:

$$\begin{aligned} Dx_1 &= x_3, \\ Dx_2 &= -x_2, \\ 0 &= x_1 + x_2 + \theta x_3, \end{aligned} \tag{1.19}$$

with singular matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.20}$$

where $\mathbf{Dx} = (Dx_1, Dx_2, Dx_3)'$.

The only equilibrium is at $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) = (0,0,0)$ For any $\theta \neq 0$, Barnett and He [2006a,b] solve the last equation for x_3 and substitute into the first equation to derive the following two equation system:

$$\begin{aligned} Dx_1 &= -\frac{x_1 + x_2}{\theta}, \\ Dx_2 &= -x_2. \end{aligned} \tag{1.21}$$

In this case, the matrix $\mathbf{B}$ becomes the identity matrix.

This two-dimensional system is stable at $\mathbf{x}^* = (x_1^*, x_2^*) = (0,0)$ for $\theta > 0$ and unstable for $\theta < 0$. However, setting $\theta = 0$, Barnett and He [2006a,b] find that system (1.19) becomes

$$\begin{aligned} x_1 &= -x_2, \\ Dx_2 &= -x_2, \\ x_3 &= x_2, \end{aligned} \tag{1.22}$$

for all $t > 0$. This system has the following singular matrix:

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.23}$$

The dimension of system (1.22) is very different from that of (1.21). In system (1.22), there are two algebraic constraints and one differential equation, while system (1.21) has two differential equations and no algebraic constraints. Clearly the matrix $\mathbf{B}$ is different in the two cases with different ranks. This example shows that singular bifurcation can results from the dependence of $\mathbf{B}$ upon the parameters, even if there does not exist a direct closed-form algebraic representation of the dependence.

Barnett and He [2008] find singularity bifurcation in their research on the Leeper and Sims' Euler-equations macroeconometric model, as surveyed in Section 3. Singularity bifurcations could similarly damage robustness of dynamic inferences with other modern Euler-equations macroeconometric models. Examples above show that implicit function systems (1.9) and (1.10) could produce singular bifurcation, while closed form differential equations systems are less likely to produce singularity bifurcation. Since Euler equation systems are in implicit function form and rarely can be solved for closed form representations, Barnett and He [2006a,b] conclude that singularity bifurcation should be a serious concern with modern Euler equations models.

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