Full text available at: http://dx.doi.org/10.1561/3100000015

HELM: The Holomorphic Embedding Load-Flow Method. Foundations and Implementations

Other titles in Foundations and Trends® in Electric Energy Systems

Sustainable Transportation with Electric Vehicles
Fanxin Kong and Xue Liu
ISBN: 978-1-68083-388-1

Unit Commitment in Electric Energy Systems
Miguel F. Anjos and Antonio J. Conejo
ISBN: 978-1-68083-370-6

Reliability Standards for the Operation and Planning of Future Electricity Networks
Goran Strbac, Daniel Kirschen and Rodrigo Moreno
ISBN: 978-1-68083-182-5

Toward a Unified Modeling and Control for Sustainable and Resilient Electric Energy Systems
Marija D. Ilic
ISBN: 978-1-68083-226-6

HELM: The Holomorphic Embedding Load-Flow Method. Foundations and Implementations

Antonio Trias
Aplicaciones en Informática Avanzada
Sant Cugat del Vallès, 08172 Spain
triast@aia.es

now
the essence of knowledge
Boston — Delft

Foundations and Trends® in Electric Energy Systems

Published, sold and distributed by:
now Publishers Inc.
PO Box 1024
Hanover, MA 02339
United States
Tel. +1-781-985-4510
www.nowpublishers.com
sales@nowpublishers.com

Outside North America:
now Publishers Inc.
PO Box 179
2600 AD Delft
The Netherlands
Tel. +31-6-51115274

The preferred citation for this publication is

A. Trias. *HELM: The Holomorphic Embedding Load-Flow Method. Foundations and Implementations*. Foundations and Trends® in Electric Energy Systems, vol. 3, no. 3-4, pp. 140–370, 2018.

ISBN: 978-1-68083-517-5

Foundations and Trends® in Electric Energy Systems

Volume 3, Issue 3-4, 2018

Editorial Scope

Topics

Foundations and Trends® in Electric Energy Systems publishes survey and tutorial articles in the following topics:

- Advances in power dispatch
- Demand-side and grid scale data analytics
- Design and optimization of electric services
- Distributed control and optimization of distribution networks
- Distributed sensing for the grid
- Distribution systems
- Fault location and service restoration
- Integration of physics-based and data-driven modeling of future electric energy systems
- Integration of Power electronics, Networked FACTS
- Integration of renewable energy sources
- Interdependence of power system operations and planning and the electricity markets
- Microgrids
- Modern grid architecture
- Power system analysis and computing
- Power system dynamics
- Power system operation
- Power system planning
- Power system reliability
- Power system transients
- Security and privacy
- Stability and control for the whole multi-layer (granulated) network with new load models (to include storage, DR, EVs) and new generation
- System protection and control
- The new stability guidelines and control structures for supporting high penetration of renewables (>50%)
- Uncertainty quantification for the grid
- System impacts of HVDC

Information for Librarians

Foundations and Trends® in Electric Energy Systems, 2018, Volume 3, 4 issues. ISSN paper version 2332-6557. ISSN online version 2332-6565. Also available as a combined paper and online subscription.

Contents

HELM: The Holomorphic Embedding Load-Flow Method. Foundations and Implementations

Antonio Trias[1]

[1] *Aplicaciones en Informática Avanzada; triast@aia.es*

ABSTRACT

The Holomorphic Embedding Load-Flow Method (HELM) was recently introduced as a novel technique to constructively solve the power flow equations in power networks, based on advanced concepts from complex analysis, algebraic curves, and modern techniques in approximation theory. In contrast to traditional methods, which rely on numerical iterative schemes whose convergence is often subject to varying degrees of uncertainty, HELM's results are always guaranteed and unequivocal: if the power flow problem is feasible, it constructs the most desirable solution; and conversely, if the power flow problem is infeasible, it signals such condition reliably. Additionally, the significance of HELM extends beyond its utilitarian role as a reliable power flow solver, since the theory backing this method is proving to be a fertile ground for the development of new analysis tools for power systems.

This work covers the HELM method from the ground up. It revisits its theoretical foundations in detail, stressing the importance of some key ideas grounded in the physics of the problem. These provide the necessary intuition for the mathematical developments to follow; in particular, for the

Antonio Trias (2018), "HELM: The Holomorphic Embedding Load-Flow Method. Foundations and Implementations", Foundations and Trends® in Electric Energy Systems: Vol. 3, No. 3-4, pp 140–370. DOI: 10.1561/3100000015.

introduction of the holomorphic embedding as a way to turn the original problem into the study of a *plane algebraic curve*, where the branches represent the power flow solutions. This is shown to be a natural way to characterize the multiple solutions to the problem, answering some deep practical questions such as: *in the absence of information about dynamic stability, which of the power flow solutions is the most desirable one for the operation of a power system?* The formulations cover both traditional ac networks and dc networks (which are gaining importance in microgrids, spacecraft, and electric aircraft). Special attention is paid to the analytic continuation of power series, in particular to the calculation of Padé approximants. It also serves to introduce the topic of higher order rational approximants, which allow reproducing the nose points around voltage collapse with better numerical stability than their Padé counterparts. An interesting by-product of this theory, *Sigma plots*, is shown to be a useful graphical tool for the quick visual assessment and diagnosis of both feasible and unfeasible cases.

Controls, such as voltage regulation by generators, are first incorporated into the method as algebraic equality constraints, with no limits in the controlling variables. The method also covers a formulation that allows for possible conflicts between the specified controls, solving them optimally. Also cover how to deal with control limits, without resorting to control type-switching approaches, presenting a novel Lagrangian formulation and using the Padé-Weierstrass (P-W) HELM method, a special analytic continuation technique that greatly increases the precision achievable with HELM.

IEEE Index Terms—Load flow, power system analysis computing, power system simulation, power system modeling, circuit analysis computing, nonlinear network analysis.

AMS 2010 Mathematics Subject Classification (MSC2010): 14H50, 14H81, 30B10, 30B40, 30B70, 30E10, 94C99.

1

Introduction

1.1 The power flow problem

Electrical power has become an essential and critical infrastructure of modern society. The power grid has been recognized by the US National Academy of Engineering as "the most influential engineering innovation of the 20th century" (Constable and Somerville, 2003). The power grid is an enormously complex network of high voltage lines, transformers, and substations that carries bulk power over long distances, from power generation facilities to distribution substations. But despite all its complexity, it is remarkable that the essential behavior of the grid can be described according to the relatively simple physical laws of electric circuits.

The cornerstone problem of electrical power systems is the so-called *power flow* (also known as *load flow*) study, which describes the steady-state of the network under some given conditions. The problem equations can be written in terms of the current balance at each bus i, as follows:

$$\sum_k Y_{ik}^{(\mathrm{tr})} V_k + Y_i^{(\mathrm{sh})} V_i = \frac{S_i^*}{V_i^*} \tag{1.1}$$

where $Y_{ik}^{(\mathrm{tr})}$ are the elements of the transmission admittance matrix, $Y_i^{(\mathrm{sh})}$ are shunt admittances, and S_i are constant-power injections going into the bus. The index k runs over all buses including the swing bus, whose voltage V_{sw} is specified as the reference. In its most basic form, the problem consists in solving (1.1) for the voltages V_i, for a given set of injections S_i. The terms appearing on the l.h.s. of the equation are all linear, while the ones on the r.h.s. are constant-power injections, which makes the problem *nonlinear and multi-valued.* A variation of this problem, closer to actual practice, takes into account the so-called PV buses, for which the voltage modulus $|V_i|$ is kept constant by means of a variable injection of reactive power Q_i provided by some generator. This amounts to adding new constraint equations together with their corresponding new variables to (1.1), as it will be shown in Section 2.

This nonlinearity and multi-valuedness is already manifest in the simplest case one can study, the two-bus model. Figure 1.1 shows schematically this model in the ac case, and its analogous counterpart in dc circuits. There is one bus containing a source that maintains the voltage fixed at a reference value V_{sw}, independently of the amount of current extracted (or absorbed). This bus receives the name of "swing" or "slack", as it provides whatever amount of power is needed to balance the system. Since the global phase angle can be arbitrary, the angle of the swing's voltage V_{sw} can be chosen at 0, without loss of generality. The swing is connected to the second bus by a transmission line with lumped impedance parameter $Z = R + jX$, or equivalently, admittance $Y = G + jB = 1/Z$. This bus has a specified constant-power injection S, which will be considered negative if it is a load, and positive if it is a generator (using the active sign convention). The power flow problem then reduces to a single equation for the bus voltage of this bus, V:

$$V = V_{\mathrm{sw}} + \frac{Z\,S^*}{V^*} \tag{1.2}$$

When working with the real and imaginary parts of V, one arrives at two equations, one of them a second degree algebraic equation. Using the shorthand notation $a \equiv \mathrm{Re}(ZS^*) = XQ + RP$ and $b \equiv \mathrm{Im}(ZS^*) =$

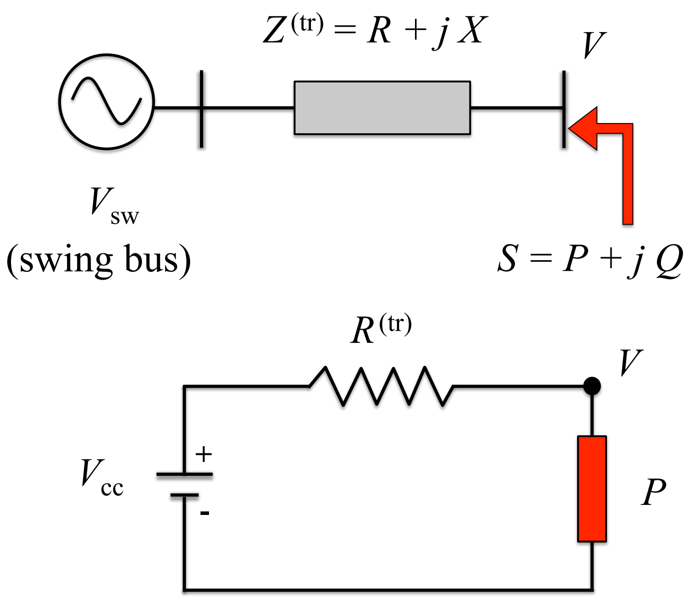

Figure 1.1: The two-bus model, in ac and dc networks.

$XP - RQ$, it is straightforward to find the solution[1]

$$V = \frac{1}{2}V_{sw} \pm \sqrt{\frac{1}{4}V_{sw}^2 + a - \frac{b^2}{V_{sw}^2}} + j\frac{b}{V_{sw}} \tag{1.3}$$

Analyzing this closed form expression it is possible to appreciate two important things. First, the nonlinearity is of a very specific kind: it is *algebraic*, i.e. it has its sources in an algebraic equation (of degree 2 in this case, for both ac and dc). Second, the multiplicity of solutions can be understood in terms of the multiple *solution branches* of said algebraic equation. Moreover, one observes how it is possible to have no solution, in other words, an infeasible power flow. This corresponds to the well-known phenomenon of voltage collapse: as the power demand increases, the system reaches a limit to the power that can be transferred across the transmission line, due to the effects of the voltage drop. Past

[1] Appendix A

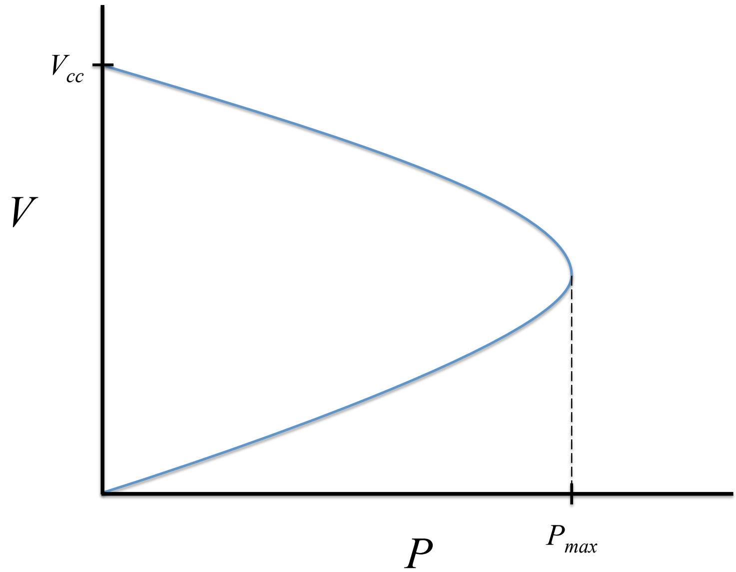

Figure 1.2: The two solution branches and voltage collapse in a dc circuit.

the point of collapse, there is no physical solution. Figure 1.2 shows schematically this phenomenon in dc systems. For ac systems, one has qualitatively the same behavior, only that both quantities V and S are complex-valued and therefore one usually depicts sections of the multi-dimensional curve $V(S)$, such as the well-known P-V or Q-V curves.

The two-bus model is perhaps underappreciated, as it brings an enormous amount of insights into the physics of the power flow problem and its mathematical structure. This claim will get substantiated in various places along this work; in particular Appendix A contains a detailed analysis that reproduces most of the essential points of HELM and the theory behind it.

Turning our attention now to the general n-bus case, it is clear that equations (1.1) are no longer solvable in closed form; they need to be solved numerically in one way or another. The following sections review the most commonly used numerical methods and analyzes their shortcomings, before introducing HELM.

1.2 An overview of iterative power flow methods

The traditional approach has been to use *iterative* numerical methods, of the kind one uses for general nonlinear problems, not specifically the power flow equations. The earliest methods were based on Gauss–Seidel (GS) iteration (Ward and Hale, 1956), which has slow convergence rates but very low memory requirements. GS may still be used when the other methods fail to converge starting from the flat profile, but most other methods are based on Newton–Raphson (NR) (Tinney and Hart, 1967), which works generally better than GS because of its quadratic convergence properties.

One of the major downsides of NR, due to the limited computing power available at the time it was introduced, was that each step involved the solution of a large linear system involving the Jacobian matrix of the system. Fortunately, the matrices arising from power systems are *very* sparse, which explains why power systems researchers were early pioneers in the area of direct numerical methods for sparse linear algebra. For instance, Tinney and Walker (Tinney and Walker, 1967) developed the first "minimum degree" reordering algorithm for reducing the fill-in in the factorization of sparse matrices, a method that is widely used today in sparse linear algebra (Davis, 2006).

Several improvements were developed such as the Fast Decoupled Load Flow (FDLF) to enhance convergence and reduce execution time which yields a good approximation in high-voltage transmission systems (Despotovic *et al.*, 1971; Stott, 1972). Decoupling leads to smaller Jacobian matrices, which can be a big computational gain in large networks. Of all the various decoupled methods based on NR, the FDLF formulation of Stott and Alsac (Stott and Alsac, 1974) has become the most successful and it is almost a de-facto standard in the industry, either in its original form or in one of its variants (Amerongen, 1989). In addition to decoupling real and reactive power, the FDLF method factorizes the Jacobian matrix only once.

To this day, most power flow methods used in commercial software for large real-world networks are still based on these numerical iterative techniques (supplemented, of course, with various additional refinements). Standard textbooks on power system analysis (Grainger and Stevenson, 1994; Kundur, 1994; Das, 2011) describe all these methods in detail. For a recent concise review, see for instance (Gómez-Expósito and Alvarado, 2009). Additionally, the Power Systems Engineering Research Center (PSERC) provides an open-source tool called MATPOWER (Zimmerman *et al.*, 2011), which contains good quality implementations of GS, NR, and FDLF (both Stott and Alsac's "XB" version and Amerongen's "BX" version).

1.3 The convergence problem of iterative methods

A common shortcoming of all these traditional methods is their unreliable and unpredictable convergence behavior. To a greater or lesser degree, numerical iteration suffers from these two pitfalls: on the one hand, there is no guarantee that the iteration will always converge, as this depends on the choice of the initial point; on the other, since the system has multiple solutions, it is not always possible to control which solution it will converge to. As it is well-known, the load flow equations have many solutions, and only one of them corresponds to the actual operating state of the electrical system. Unless one provides a starting point sufficiently close to the desired solution, iterative schemes may not just fail to converge, but converge to a different one.

The root causes of these convergence problems are deep. At the core of any method based on numerical iteration we find the idea of a map that, when iterated, is expected to converge to a fixed point. One seeks maps having the *contraction* property, which in principle guarantees that the iteration will converge to a fixed point starting from any point belonging to a certain (non-empty) set. Such set is called the *basin of attraction* of said fixed point under the map. However, it is well known that these basins of attraction are impossible to characterize with precision, as their borders are *fractal*, in general (McDonald *et al.*, 1985). This problem is particularly relevant for power systems, where the power flow equations contain multiple solutions, each with their own basin of attraction under the iterative scheme. Several authors have explored this fractality problem in the context of power flow (DeMarco and Overbye, 1988; Thorp and Naqavi, 1989; Thorp and Naqavi, 1997; Klump and Overbye, 2000a; Mori, 2000), showing with numerical experiments how the borders between neighboring basins intertwine in very complex patterns, also interspersed with points that lead to divergence. Figure 1.3 shows an example calculated on the IEEE-300 test case, using the Newton–Raphson method. The test case has been loaded at bus 528 in order to stress the system (i.e. bring it closer to a voltage collapse point), and therefore reveal the fractality problem more clearly.

While it is true that in practice the basins of attraction are usually large enough to allow an experienced engineer to find a good initial point, the key point to stress here is that there exists no general characterization for the shape and size of those basins[2]. Therefore, *there is no general procedure to ensure convergence to the desired solution in a completely unattended fashion, unsupervised by human experts*. HELM was actually born out of the need to develop new software applications for real-time decision-support in transmission operation. There, the algorithms rely on performing a massive number of exploratory power flows, many of them corresponding to abruptly changing scenarios. These algorithms cannot afford that a percentage of cases, however small, had a chance of diverging or mis-converging to undesired solutions. The same thing would happen to any sort of autonomous control, if it needed to rely on power flow calculations.

Of course, many approaches have been developed to minimize the chances of unpredictable behavior in iterative methods—but to our knowledge, none of them can guarantee convergence in an automated, unsupervised fashion. Some efforts have been made to understand and characterize the regions of convergence (Wu, 1977), with very limited success. Many heuristics have been devised to come up with better starting seeds (Leonidopoulos, 1995; Klump and Overbye, 2000b;

[2]Specifically for Newton–Raphson, the Kantorovich theorem does in principle provide a criteria to find out whether a given starting point will converge or not, but this is unusable in practice as it requires computing the Lipschitz constant of the Jacobian (at the chosen initial guess), which is highly impractical.

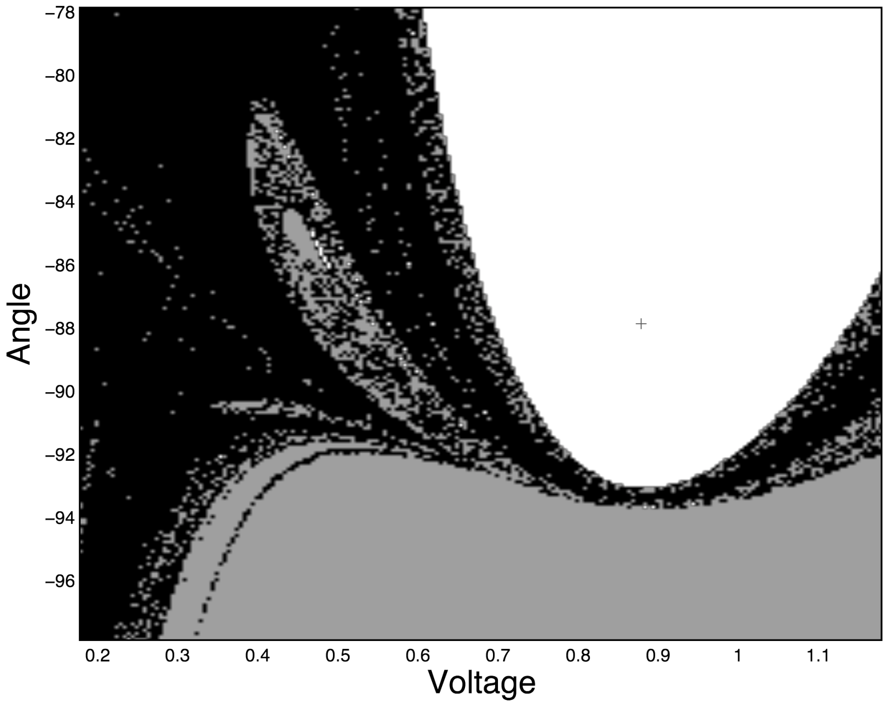

Figure 1.3: Fractal basins of attraction in a stressed IEEE-300 test case (at bus 528). The area in white is the basin of attraction of the "high-voltage" solution, which usually corresponds to the actual operating state of the network (the coordinates of this solution are marked with a cross); the gray areas are the basin of attraction of all other solutions, which contain one or more buses in a low-voltage state; and the black areas are the basin of non-convergence.

Murray *et al.*, 2013), which helps to mitigate non-convergence in some cases. Another common mitigation strategy consists in monitoring and limiting the size of each Newton–Raphson step, usually with some criteria to find the best fraction of the step along the Newton direction (see (Deuflhard, 2011); and also (Press *et al.*, 2007), Chapter 9.7). This is the idea behind Iwamoto and Tamura's "optimal multiplier" method (Iwamoto and Tamura, 1981) and its many descendants. Reference (Schaffer and Tylavsky, 1988) provides a good review of these variants, and (Braz *et al.*, 2000; Tate and Overbye, 2005) provide more recent assessments and comparisons between these methods. Note that the term "globally convergent", sometimes used in relation to these step-size changing methods, is a misnomer: there is really no guarantee of convergence, let alone control on what solution is selected.

The methods that have the best chance of avoiding convergence problems are those based on *homotopy*, also known as continuation or path-following methods. The most commonly known is the Continuation Power Flow (CPF) of Ajjarapu and Christy (Ajjarapu and Christy, 1992), but there are also many modern implementations; see for instance (Mehta *et al.*, 2016b) and the review in (Mehta *et al.*, 2016a). These methods are discussed in more detail later on in Section 1.6, touching upon the similarities and differences with HELM. Continuation methods are typically much slower, but the point here is that convergence still remains a potential problem. And since convergence is not 100% ensured in all cases, human supervision is still needed to assess the results obtained.

1.4 Non-iterative methods

As evidenced above, there is a need for a direct, fully reliable load flow method. One theoretical possibility is solving the algebraic equations exactly, in closed form. This can be done because they are actually a set of polynomial equations in the voltage variables, and these can be solved using polynomial elimination techniques (resultants and Gröbner Basis) with the help of computer algebra packages (Montes, 1998; Ning *et al.*, 2009; Mehta *et al.*, 2016a). However, this has very strong limitations, since the memory and computational costs increase exponentially (or faster) with network size, making it impossible to get past 5 or 6 buses even with the most capable computers available today[3]. For the foreseeable future, even power flows with only hundreds of buses are out of reach for these computer-algebra approaches, unless there is a fundamental breakthrough. There is however, great value in being able to explore closed-form solutions for, say, 3 or 4 bus problems, as there is a lot to be gained in terms of mathematical intuition into the interplay of all solution branches.

[3]At least when using the most powerful desktop computers available as of 2018, and computer algebra packages such as Maple.

A more interesting approach from the practical point of view is the so-called Series Load Flow (Xu *et al.*, 1998; Zambroni De Souza *et al.*, 2007), based on an earlier idea by Sauer (Sauer, 1981). This method uses the Taylor expansion of the voltage variables as functions of all the specified parameters of the problem (power injections, voltage magnitudes), calculated on a point at which the solution is known. Summation of the Taylor series then allows to extend the solution to other scenarios, in a process that is limited by the radius of convergence of the series. Although it was developed completely independently, the Holomorphic Embedding method is somewhat related to these ideas, but with one key difference: the Series Load Flow uses real variables and therefore cannot guarantee convergence of the Taylor series in general, for arbitrary ranges. By contrast, the Holomorphic Embedding Load Flow is based on Complex Analysis. This seemingly minor technicality makes an enormous difference. It is only by working in the complex field and using the wonderful properties of algebraic curves and holomorphic functions that the method achieves its desired properties of completeness (i.e., give the right solution when it exists; and unambiguously signal infeasibility when it does not). Additionally, the approach of real-valued Taylor series does not offer any new analytic insights, other than those very specific to the particular study case. HELM, on the other hand, brings along a new set of tools and ideas offering new theoretical insights into the general problem of power flow.

1.5 HELM and its significance

The Holomorphic Embedded Load Flow was first presented in (Trias, 2012), after being awarded two US patents for its industrial applications (Trias, 2009; Trias, 2011). This method radically broke away from the established iterative methods by employing a whole different set of techniques. Its most defining features are that it is non-iterative, constructive (and therefore deterministic), and provides *unequivocal results*: backed by a mathematical proof, it obtains the correct solution to the multivalued load flow problem when it is feasible), and otherwise signals the non-existence of a solution when the problem is infeasible (i.e., beyond a point of voltage collapse).

The method is based on a *holomorphic embedding* that turns the voltage variables into analytic functions in the complex plane. In doing so, the power flow problem is converted from a set of algebraic equations into a much richer structure: a problem in *plane algebraic curves*. This provides a framework to study and obtain solutions using the full power of algebraic geometry and complex analysis techniques.

HELM's core procedure consists in the construction of the power series of complex voltages at a well-defined reference point of the embedding, where it is trivial to identify the correct branch of the multivalued problem (the so-called "reference state"). It then uses analytic continuation by means of Padé approximants to reach the objective. This last step is backed by Stahl's theory (Stahl, 1985a; Stahl, 1985b; Stahl, 1989; Stahl, 1997), and it is what confers the completeness properties upon the method: since the Padé approximants (actually, certain sequences on the Padé table) are proven to converge to the voltage functions in their maximal domain of analyticity, the method obtains the solution when it exists, and signals infeasibility when it does not. Section 2 provides the rigorous details.

From a purely numerical standpoint, one can see that HELM's paradigm is that of Approximation Theory. In contrast to numerical iterative schemes with unreliable and hard to characterize convergence behavior, with HELM one constructs successive approximations, guaranteed to reach the desired precision if one is willing to put the necessary effort into it (essentially: the number of terms in the series, and dealing with finite-precision arithmetic). This is the way many special functions (trigonometrics, exponential, Bessel, ...) are actually calculated in pocket calculators and computers, whenever Newton–Raphson and other fixed-point iterative schemes just do not have the convergence guarantees to be *reliable*[4]. In this respect, probably the single most important impact of HELM is that it *enables* reliable, real-time, intelligent applications. The HELM method was actually born out of the need for a fully reliable load flow in the context of some AI-based applications that depend critically on the ability to perform exploratory load flow

[4]To our knowledge, only division and the real square root are sometimes implemented via Newton–Raphson, because in those elementary cases the convergence properties *are* guaranteed.

studies, with absolutely no margin for failure. The most prominent examples are two decision-support tools, a *Limits Violation Solver* and a *Restoration Plan Builder*, implemented successfully in an industrial-strength EMS (AIA, 1998–2017). These tools are fully model-based thanks to a technique well-known in the AI community: searching for optimal paths in the state-space of the system, using the A^* algorithm. In this case the state-space consists of all possible electrical (steady) states that the network can achieve, and the available SCADA actions provide transitions between them. The algorithm also needs sophisticated heuristics to guide the search efficiently, but the load flow method needs to be 100% reliable, since it is used at each and every step of the exploration. Of course, other real-time tools such as Contingency Analysis, or the online calculation of PV/QV Curves also benefit from increased reliability.

But another promising potential of the method lies in the *new insights* it brings into the analysis of the load flow problem. The treatment in terms of algebraic curves could prove quite powerful. For instance, it provides a coherent framework for the characterization and computation of all the multiple solutions to the original problem in the radial system case (white, black, ghost solutions). Given the vast amounts of results in the field of algebraic curves in Complex Analysis, it is reasonable to think that HELM is just scratching the surface of what is potentially possible. The theory of approximants (rational or other) is another source for insights and practical results. For instance, the zeros and poles of Padé approximants tend to accumulate on Stahl's (minimal) set of branch cuts of the functions $V(s)$ (see (Baghsorkhi and Suetin, 2015; Baghsorkhi and Suetin, 2016a; Baghsorkhi and Suetin, 2016b) for several examples). Therefore their values, or even their evolving patterns as the approximant order increases, may be used as new indicators, in ways that have not been fully explored yet. To sum up, one may think of HELM theory as a new language for the analysis of an old problem.

1.6 Conceptual differences with continuation methods

The apparent similarities between HELM and continuation methods warrant an extended discussion in order to clarify some quite important points. It is worthwhile to clear up this common misunderstanding as soon as possible: the holomorphic embedding method has absolutely nothing to do with the Continuation Power Flow (CPF) and other homotopy-based methods. In order to fully appreciate the profound differences that separate these, one must first understand the key concepts behind homotopy.

Homotopic continuation (Allgower and Georg, 2003) is a broad, general-purpose mathematical technique that addresses nonlinear problems. Homotopy-based methods are also known as “path-following” methods. In the domain of ac power flow, homotopy is the basis for several methods specifically devised to address convergence problems of iterative power flows. The most widely known is the Continuation Power Flow (CPF) (Ajjarapu and Christy, 1992; Zaborsky and Ilić, 2000), but there are also many refined variants; see for instance the recent review in (Mehta *et al.*, 2016a). Homotopy is also used extensively in the domain of electronics, for finding the dc operating point (or points) of general nonlinear circuits (Wolf and Sanders, 1996; Trajković, 1999; Trajković, 2012).

Like HELM, continuation methods are also based on the general idea of the *embedding* technique, whereby one is able to solve the problem at the “easy” limit of the embedding parameter ($\lambda = 0$) and then follow this solution up to the value of the parameter where the original system is recovered ($\lambda = 1$). However, the key difference is that *homotopy only exploits continuity and single differentiability*. It is therefore a local method, essentially. It only requires that the starting point satisfies the conditions of the Implicit Function Theorem (i.e. no degeneracy at $\lambda = 0$), and that no bifurcations are encountered on the path up to $\lambda = 1$. Certain bifurcations, namely turning points of the curve with respect to the parameter, may be overcome by switching to arc-length parametrization. In the context of finding dc operating points of nonlinear circuits, probability-1 globally convergent continuation methods (Melville *et al.*, 1993; Wolf and Sanders, 1996) ensure that no bifurcations will be encountered (although there is no control as to what solution the homotopy arrives at). For these curve-tracking calculations, methods use either ODE-integration techniques or a predictor-corrector scheme, where the corrector is based on Newton–Raphson or quasi-Newton iteration.

Modern continuation methods are considered to be computationally slow but robust; however, their reliance on numerical iteration may still cause problems in practice (Zaborsky and Ilić, 2000; Roychowdhury and Melville, 2006). The fractality problem remains, because no numerical iterative method can guarantee the correct convergence in *all* possible cases.

Related also to homotopic continuation, but coming from a very different domain, the Homotopic Analysis Method of Liao (Liao, 1992; Liao, 2003; Liao, 2013) does share more commonalities with HELM. The method targets nonlinear ordinary and partial differential equations (ODE/PDE) in fluid mechanics and several other areas of applied mathematics (Liao, 1999; Vajravelu and Van Gorder, 2013). It uses Taylor series expansions, and it computes the coefficients by a procedure completely analogous to HELM's N-th order representation, transforming the nonlinear problem into an infinite sequence of linear ones just like HELM. The method even uses Padé approximants as well, for accelerating the sum of the power series. However, this method only uses real analysis techniques, so the same limitations discussed about the Series Load Flow in Section 1.4 apply here. In particular, the Padé approximants are not guaranteed to converge, in general.

In contrast to homotopic continuation, the holomorphic embedding technique exploits *holomorphism*, in other words, *complex analyticity*. This is a much stronger condition than single differentiability. In fact, holomorphism endows the method with *global* properties, because knowledge of the power series at a given point (that is, all its derivatives) can be used to reconstruct the function everywhere, by virtue of analytic continuation. It should be emphasized that analytic continuation and homotopic continuation are two different mathematical topics that have nothing in common. Analytic continuation is a technique widely used in complex analysis to extend the domain of a given holomorphic function. In particular, for holomorphic functions given as a power series representation, it is typically used to extend the function beyond the radius of convergence of the series. For a general mathematical problem, it is not known *a priori* what analytic continuation technique can be successful (other than a cumbersome re-evaluation of the power series at a different point), or what will be the extent of the maximal domain to which the function can be analytically extended. But, in the context of the power flow problem, the HELM method does provide a method to perform analytic continuation, and moreover, it ensures that it is *maximal*. This analytic continuation is provided by the sequence of near-diagonal Padé approximants constructed from the power series, as established by Stahl's theorems. As a way of showing the profound differences with homotopic continuation without delving any further in the mathematical foundations (see Section 2 and (Trias, 2015)), a few practical differences can be pointed out: HELM is able to *directly* calculate the solution at any point along the embedded path (actually, the complex plane), without needing any previous point. Moreover, the calculation is carried out by a constructive procedure having a well-characterized convergence behavior. To conclude this discussion on the conceptual differences, a few remarks must be made on another common misunderstanding about the holomorphic embedding method, regarding its use of power series and "approximants". In contrast to real analysis, power series play a major role in complex analysis, and even more so in the field of algebraic curves. Actually, power series are one of the major ways of *representing* holomorphic functions. Far from being an approximation, the power series *is* the function. And although the power series only converges within its radius of convergence, analytic continuation allows one to make use of the information contained in the power series to reconstruct the function well beyond that radius. Note that, for general holomorphic functions, there is no known procedure to obtain the maximal analytic continuation, but herein lies the power of Stahl's theorem: it states that, for the class of functions HELM deals with, the near-diagonal sequences of Padé approximants converge to the function in their maximal logarithmic capacity domain of analytic continuation. Therefore, the method is not limited by any convergence issues (within the limits of finite-precision arithmetic), because the successive Padé approximants can get as close as desired to the sought solution—one just needs to obtain more terms of the power series. And, as shown in Section 5, even when finite-precision arithmetic is an issue, there is a well-defined way to avoid it.

1.7 An outline of this work

This work is structured as follows. Section 2 focuses on the foundational concepts and the theory on which HELM is grounded. It starts by reviewing the HELM method centered on the core procedures. Then it goes deep into the foundations by establishing the underlying mathematical structure of the theory, which sits on two pillars: one algebraic-geometric; the other, complex-analytic. The algebraic-geometric view stems from a fundamental projective invariance of the power flow equations. This invariance has deep, physically interpretable roots, and it is shown how it is an essential guide for devising embeddings that ultimately allow treating the power flow problem as essentially a study in algebraic curves. Of course, a non-trivial technical point is that such embeddings need to be holomorphicity-preserving (for this, the introduction of the $\widehat{V}(s)$ functions plays a key role in getting rid of the complex conjugation). Algebraic curves also help to shed light on the physical characterization of the multiple solutions of the powerflow problem, showing why HELM's solution (the so-called white branch) is the most desirable one for the operation of power systems. Complementing this algebraic-geometric viewpoint, it is shown how to apply standard techniques of Complex Analysis (power series) for practical computation of the solution. Stahl's theorem on the maximality of the analytic continuation provided by Padé approximants then ensures the completeness of the method.

The treatment of controls is introduced by showing how to include the most common kind, namely voltage control on a local bus by means of synchronous generators (what is commonly referred to as a PV-type bus). Here an emphasis is made on the fact that controls can be easily incorporated into HELM as long as they can be expressed as additional *algebraic constraints* for the power flow equations, in which case holomorphicity is preserved and the problem can still be described in terms of an algebraic curve. Of course, each new constraint brings along one new variable into the system (e.g., the reactive injection Q_i in case of PV buses), so as to keep the balance between the total number of equations and unknowns in the system. Since HELM somehow converts the original system of nonlinear equations into an infinite sequence of linear ones (in what will be referred to as the *"N-th order* *representation"*), the treatment of controls is an excellent opportunity to show how linear algebra manipulations can be used to our advantage. For instance, it is shown how PV buses can actually reduce the number of equations and unknowns in the system, by using transformations that simply amount to a form of Gaussian elimination. A more advanced example of the kind of new possibilities made available in the N-th order representation is the treatment of generalized controls, where it is shown how to deal with *conflicting* controls via the singular value decomposition (SVD). Control *limits*, however, need a whole different approach, and their treatment is deferred until Section 5. The section concludes with a discussion on how the method is extended to accommodate "smooth" controls, such as the ubiquitous generator-controlled PV bus.

In Section 3, the method is extended from ac to fully-dc power systems, which is of interest in the area of modern autonomous micro-grids for terrestrial uses, spacecraft, and more-electric ship and aircraft. Through an appropriate embedding technique, the method is shown to extend naturally to dc power transmission systems, preserving all the constructive and deterministic properties it has in the ac world. However, these systems are characterized by the presence of multiple power-electronics devices with highly nonlinear behavior, so the implementation of HELM for some of these devices may not be entirely evident at first. Therefore this section shows a few non-trivial examples, such as a photovoltaic array feeding a constant-power load. The extension to the general problem of finding dc operating points in electronics is also discussed, and exemplified on the diode model.

Section 4 is dedicated to the topic of Padé Approximants and other analytic continuation techniques of interest in HELM. Since this is a quite vast and very well developed area of applied mathematics and numerical analysis, the exposition will not try to develop any specific numerical method for computing approximants. Instead, the discussion will offer the most relevant references and pointers to practical implementations, typical caveats to observe, limitations, and advice for would-be implementors. Perhaps more interestingly from the conceptual point of view is the exposition on higher-order algebraic approximants (Padé-Hermite), which, when viewed under the light of an *osculation* of the underlying algebraic curve, can be used as reduced pseudo-equivalents.

In particular, one class of higher-order approximants that has found interesting practical applications is the *Sigma approximant*, which provides novel diagnostic capabilities in a very visual way, especially for infeasible cases (where a non-converging iterative method would be silent).

Section 5 covers a quite recent development in the theory of HELM: how to deal with control limits, without resorting to control type-switching or any other sort of "outer loop" approaches, presenting a novel Lagrangian formulation, arguably the one that is closest to the physics of transmission networks. The method has been termed *Padé-Weierstrass* (P-W) HELM, after the special analytic continuation technique it uses (which, as an added benefit, greatly increases the precision achievable with HELM at nose points).

Section 6 ends with a wrap-up of this work, and points to several possible avenues for future research in HELM and HELM-related methods.

Some interesting topics, whose exposition is more self-contained, have been relegated to the appendices. Appendix A is an end-to-end application of all the theoretical concepts given in the main text to the two-bus model. The advantage of doing so is that one can work out all results in closed form, and gain an enormous amount of intuition that most of the times (but not all) carries over to the n-bus case with good fidelity. Appendix B then presents a few assorted numerical results, starting with the rate of convergence of Padé approximants in several cases and load/generation scenarios. Then it is shown how full precision can be recovered thanks to the new P-W method.

In the Appendix C the convergence of the continuous fraction is extended for complex s values. The HELM P-W method is summarized in Appendix D and finally Appendix E presents a method (Trias, 2017a) that allows finding *all* power flow solutions, by focusing on the study of the whole algebraic curve, i.e. all of its branches. The method only applies to networks with a very special topology, but it is remarkable because it effectively achieves the full elimination procedure.

Throughout this work, the author has also tried to do their best job at citing all papers from other authors on HELM and Holomorphic Embedding-related methods. Although the subject is still young, we have found that there are already many quality papers published on the subject. We have done our best to identify all these before this manuscript went to press.

References

Ahlfors, L. V. 1979. *Complex analysis: an introduction to the theory of analytic functions of one complex variable*. McGraw-Hill.

AIA, E. 1998–2017. "AGORA". URL: http://elequant.com/products-and-solutions/agora.

Ajjarapu, V. and C. Christy. 1992. "The continuation power flow: a tool for steady-state voltage stability analysis". *IEEE Trans. Power Syst.* 7(1): 416–423. DOI: 10.1109/59.141737.

Allan, R. N. and C. Arruda. 1982. "LTC Transformers and MVAr Violations in the Fast Decoupled Load Flow". *IEEE Trans. Power App. Syst.* PAS-101(9): 3328–3332. DOI: 10.1109/TPAS.1982.317591.

Allgower, E. L. and K. Georg. 2003. *Introduction to numerical continuation methods. Classics in applied mathematics*. SIAM.

Amerongen, R. A. M. van. 1989. "A general-purpose version of the fast decoupled load flow". *IEEE Trans. Power Syst.* 4(2): 760–770. DOI: 10.1109/59.193851.

Anderson, E., Z. Bai, C. Bischof, S. Blackford, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, and D. Sorensen. 1999. *LAPACK Users' Guide*. Third. Philadelphia, PA: Society for Industrial and Applied Mathematics. ISBN: 0-89871-447-8 (paperback).

Baghsorkhi, S. S. and S. P. Suetin. 2015. "Embedding AC Power Flow with Voltage Control in the Complex Plane: The Case of Analytic Continuation via Padé Approximants". *arXiv e-prints*. abs/1504.03249. arXiv: 1504.03249 [cs.SY].

Baghsorkhi, S. S. and S. P. Suetin. 2016a. "Embedding AC Power Flow in the Complex Plane Part I: Modelling and Mathematical Foundation". *arXiv e-prints*. abs/1604.03425. arXiv: 1604.03425 [cs.SY].

Baghsorkhi, S. S. and S. P. Suetin. 2016b. "Embedding AC Power Flow in the Complex Plane Part II: A Reliable Framework for Voltage Collapse Analysis". *arXiv e-prints*. abs/1609.01211. arXiv: 1609.01211 [cs.SY].

Baker, G. A. and P. R. Graves-Morris. 1996. *Padé Approximants*. *Encyclopedia of mathematics and its applications*. Cambridge University Press.

Basiri-Kejani, M. and E. Gholipour. 2017. "Holomorphic Embedding Load-Flow Modeling of Thyristor-Based FACTS Controllers". *IEEE Trans. Power Syst.* 32(6): 4871–4879. DOI: 10.1109/TPWRS.2017.2682117.

Bauer, F. L. 1959. "The quotient-difference and epsilon algorithms". In: *On numerical approximation*. Ed. by R. E. Langer. University of Wisconsin Press. 361–370.

Beckermann, B. and G. Labahn. 2000. "Effective computation of rational approximants and interpolants". *Reliab. Comput.* 6(4): 365–390.

Bergen, A. R. and V. Vittal. 2000. *Power Systems Analysis*. 2nd edition. Prentice Hall.

Black, A. N. and R. V. Southwell. 1938. "Relaxation Methods Applied to Engineering Problems". *Proc. Royal Soc. A*. 164(919): 447–467. DOI: 10.1098/rspa.1938.0030.

Bojanczyk, A. W., R. P. Brent, F. R. De Hoog, and D. R. Sweet. 1995. "On the Stability of the Bareiss and Related Toeplitz Factorization Algorithms". *SIAM J. Matrix Anal. Appl.* 16(1): 40–57. DOI: 10.1137/S0895479891221563.

Braz, L. M. C., C. A. Castro, and C. A. F. Murati. 2000. "A critical evaluation of step size optimization based load flow methods". *IEEE Trans. Power Syst.* 15(1): 202–207. DOI: 10.1109/59.852122.

Brent, R. P. 1989. “Old and new algorithms for Toeplitz systems”. In: *Proceedings SPIE*. Vol. 975. 2–9.

Brent, R. P., F. G. Gustavson, and D. Y. Y. Yun. 1980. “Fast solution of Toeplitz systems of equations and computation of Padé approximants”. *J. Algorithms*. 1(3): 259–295.

Bruno, O. P. and F. Reitich. 1994. “Approximation of analytic functions: a method of enhanced convergence”. *Math. Comput.* 63(207): 195–213. DOI: 10.1090/S0025-5718-1994-1240654-9.

Buchberger, B. and F. Winkler. 1998. *Gröbner bases and applications*. Vol. 251. *London Mathematical Society Lecture Note Series*. Cambridge University Press.

Cabay, S. and R. Meleshko. 1993. “A weakly stable algorithm for Padé approximants and the inversion of Hankel matrices”. *SIAM J. Matrix Anal. Appl.* 14(3): 735–765.

Chang, S.-K. and V. Brandwajn. 1988. “Adjusted solutions in fast decoupled load flow”. *IEEE Trans. Power Syst.* 3(2): 726–733. DOI: 10.1109/59.192928.

Chang, S.-K. and V. Brandwajn. 1991. “Solving the adjustment interactions in fast decoupled load flow”. *IEEE Trans. Power Syst.* 6(2): 801–805. DOI: 10.1109/59.76728.

Chen, T. and D. Mehta. 2017. “On the Network Topology Dependent Solution Count of the Algebraic Load Flow Equations”. *IEEE Trans. Power Syst.* PP(99): 1–1. DOI: 10.1109/TPWRS.2017.2724030.

Chiang, H. D. 1989. “Study of the existence of energy functions for power systems with losses”. *IEEE Trans. Circuits Syst.* 36(11): 1423–1429. DOI: 10.1109/31.41298.

Chiang, H. D. 2011. *Direct Methods for Stability Analysis of Electric Power Systems*. Wiley. DOI: 10.1002/9780470872130.

Chiang, H. D., T. Wang, and H. Sheng. 2017. “A Novel Fast and Flexible Holomorphic Embedding Method for Power Flow Problems”. *IEEE Trans. Power Syst. (to appear)*. DOI: 10.1109/TPWRS.2017.2750711.

Cho, B. H., J. R. Lee, and F. C. Y. Lee. 1990. “Large-signal stability analysis of spacecraft power processing systems”. *IEEE Trans. Power Electron.* 5(1): 110–116.

Christie, R. D. “Power Systems Test Case Archive”. University of Washington. URL: https://www.ee.washington.edu/research/pstca/.

Chua, L. O., C. A. Desoer, and E. S. Kuh. 1987. *Linear and Nonlinear Circuits*. McGraw-Hill.

Constable, G. and B. Somerville. 2003. *A Century of Innovation: Twenty Engineering Achievements that Transformed Our Lives*. Joseph Henry Press. DOI: 10.17226/10726. URL: http://www.nap.edu/catalog/10726/a-century-of-innovation-twenty-engineering-achievements-that-transformed-our.

Crutchley, D. and M. Zwolinski. 2003. "Globally convergent algorithms for DC operating point analysis of nonlinear circuits". *IEEE Trans. Evol. Comput.* 7(1): 2–10.

Das, J. C. 2011. *Power System Analysis: Short-Circuit Load Flow and Harmonics*. 2nd Edition. *Power Engineering (Willis)*. CRC Press.

Davis, T. A. 2006. *Direct methods for sparse linear systems*. Philadelphia: SIAM. DOI: 10.1137/1.9780898718881.

Davis, T. A. and E. Palamadai Natarajan. 2010. "Algorithm 907: KLU, A Direct Sparse Solver for Circuit Simulation Problems". *ACM Trans. Math. Softw.* 37(3): 1–17. DOI: 10.1145/1824801.1824814.

DeMarco, C. L. and T. J. Overbye. 1988. "Low voltage power flow solutions and their role in exit time based security measures for voltage collapse". In: *27th IEEE Conference on Decision and Control*. 2127–2131. DOI: 10.1109/CDC.1988.194709.

Despotovic, S. T., B. S. Babic, and V. P. Mastilovic. 1971. "A Rapid and Reliable Method for Solving Load Flow Problems". *IEEE Trans. Power App. Syst.* PAS-90(1): 123–130. DOI: 10.1109/TPAS.1971.292949.

Deuflhard, P. 2011. *Newton Methods for Nonlinear Problems: Affine Invariance and Adaptive Algorithms*. Vol. 35. *Springer Series in Computational Mathematics*. Springer.

Dobson, I. and L. Lu. 1992. "Voltage collapse precipitated by the immediate change in stability when generator reactive power limits are encountered". *IEEE Trans. Circuits Syst. I.* 39(9): 762–766. DOI: 10.1109/81.250167.

Dong, F., T. Kostyniak, and B. Lam. 2012. "Dealing with power flow solution difficulties". *Siemens PTI eNewsletters*. 111(Mar.): 1–3. URL: http://w3.usa.siemens.com/smartgrid/us/en/transmission-grid/products/grid-analysis-tools/pages/enewsletter.aspx.

Farkas, H. M. and I. Kra. 2012. *Riemann Surfaces. Graduate Texts in Mathematics*. Springer New York.

Fedorov, S. I. 1985. "On a variational problem of Chebotarev in the theory of capacity of plane sets and covering theorems for univalent conformal mappings". *Math. USSR-Sb.* 52(1): 115–133. DOI: 10.1070/SM1985v052n01ABEH002880.

Feng, Y. 2015. "Solving for the Low-Voltage/Large-Angle Power-Flow Solutions by Using the Holomorphic Embedding Method". *PhD thesis*. Arizona State University.

Fischer, G. 2001. *Plane Algebraic Curves*. Vol. 15. *Student Mathematical Library*. American Mathematical Society.

Fliscounakis, S., P. Panciatici, F. Capitanescu, and L. Wehenkel. 2013. "Contingency Ranking With Respect to Overloads in Very Large Power Systems Taking Into Account Uncertainty, Preventive, and Corrective Actions". *IEEE Trans. Power Syst.* 28(4): 4909–4917. DOI: 10.1109/TPWRS.2013.2251015.

Gamelin, T. W. 2001. *Complex analysis*. Springer.

Gómez-Expósito, A. and F. L. Alvarado. 2009. "Load Flow". In: *Electric Energy Systems: Analysis and Operation*. Ed. by A. Gómez-Expósito, A. J. Conejo, and C. Cañizares. CRC Press. 95–126.

Gonnet, P., S. Güttel, and L. N. Trefethen. 2013. "Robust Padé approximation via SVD". *SIAM Rev.* 55(1): 101–117.

Gragg, W. B. 1972. "The Padé table and its relation to certain algorithms of numerical analysis". *SIAM Rev.* 14(1): 1–62.

Grainger, J. J. and W. D. Stevenson. 1994. *Power system analysis. McGraw-Hill series in electrical and computer engineering: Power and energy*. McGraw-Hill.

Henrici, P. 1993. *Applied and Computational Complex Analysis, Vols 1–3. Wiley Classics Library (Book 43)*. Wiley-Interscience. ISBN: 9780471598923.

Herlihy, M. and N. Shavit. 2012. *The Art of Multiprocessor Programming, Revised Reprint*. Elsevier Science.

Hiskens, I. A. and B. B. Chakrabarti. 1996. "Direct calculation of reactive power limit points". *Int. J. Elec. Power.* 18(2): 121–129. DOI: 10.1016/0142-0615(95)00058-5.

Irisarri, G. D., X. Wang, J. Tong, and S. Mokhtari. 1997. "Maximum loadability of power systems using interior point nonlinear optimization method". *IEEE Trans. Power Syst.* 12(1): 162–172. DOI: 10.1109/59.574936.

Iwamoto, S. and Y. Tamura. 1981. "A Load Flow Calculation Method for Ill-Conditioned Power Systems". *IEEE Trans. Power App. Syst.* PAS-100(4): 1736–1743. DOI: 10.1109/TPAS.1981.316511.

Jeans, J. H. 1933. *The Mathematical Theory of Electricity and Magnetism.* 5th Edition. Cambridge University Press.

Johnson, B. K. 1977. "Extraneous and false load flow solutions". *IEEE Trans. Power App. Syst.* 96(2): 524–534. DOI: 10.1109/T-PAS.1977.32362.

Josz, C., S. Fliscounakis, J. Maeght, and P. Panciatici. 2016. "AC Power Flow Data in MATPOWER and QCQP Format: iTesla, RTE Snapshots, and PEGASE". *arXiv e-prints.* abs/1603.01533. arXiv: 1603.01533 [math.OC].

Joukowsky, N. 1910. "Über die Konturen der Tragflächen der Drachenflieger". *Zeitschrift für Flugtechnik und Motorluftschiffahrt.* 1: 281–284 and (1912) 3: 81–86.

Jovcic, D., D. Van Hertem, K. Linden, J.-P. Taisne, and W. Grieshaber. 2011. "Feasibility of DC transmission networks". In: *Innovative Smart Grid Technologies (ISGT Europe).* IEEE. 1–8.

Kataoka, Y. 2005. "A smooth power flow model of electric power system with generator reactive power limits taken into consideration". In: *2005 IEEE International Symposium on Circuits and Systems.* Vol. 5. 5286–5289. DOI: 10.1109/ISCAS.2005.1465828.

Kataoka, Y. and Y. Shinoda. 2005. "Voltage stability limit of electric power systems with Generator reactive power constraints considered". *IEEE Trans. Power Syst.* 20(2): 951–962. DOI: 10.1109/TPWRS.2005.846080.

Kavasseri, R. G. and P. Nag. 2008. "An Algebraic Geometric Approach to Analyze Static Voltage Collapse in a Simple Power System Model". In: *Fifteenth National Power Systems Conference (NPSC).* IIT Bombay. 482–487. URL: https://www.ee.iitb.ac.in/student/~npsc2008/NPSC_CD/Data/Oral/EIC2/p55.pdf.

Kendig, K. 2011. *A Guide to Plane Algebraic Curves. Dolciani Mathematical Expositions*. Mathematical Association of America.

Klump, R. P. and T. J. Overbye. 2000a. "A new method for finding low-voltage power flow solutions". In: *Power Engineering Society Summer Meeting*. Vol. 1. IEEE. 593–597. DOI: 10.1109/PESS.2000.867653.

Klump, R. P. and T. J. Overbye. 2000b. "Techniques for improving power flow convergence". In: *Power Engineering Society Summer Meeting*. Vol. 1. IEEE. 598–603. DOI: 10.1109/PESS.2000.867654.

Korsak, A. J. 1972. "On the Question of Uniqueness of Stable Load-Flow Solutions". *IEEE Trans. Power App. Syst.* PAS-91(3): 1093–1100. DOI: 10.1109/TPAS.1972.293463.

Kundur, P. 1994. *Power system stability and control. EPRI power system engineering series*. McGraw-Hill Education.

Kunz, E. and R. Belshoff. 2007. *Introduction to Plane Algebraic Curves*. Birkhäuser Boston.

Leonidopoulos, G. 1995. "Approximate linear decoupled solution as the initial value of power system load flow". *Electr. Power Syst. Res.* 32(3): 161–163. DOI: 10.1016/0378-7796(94)00911-M.

Li, T.-Y., T. Sauer, and J. A. Yorke. 1987. "Numerical Solution of a Class of Deficient Polynomial Systems". *SIAM J. Numer. Anal.* 24(2): 435–451. DOI: 10.1137/0724032.

Liao, S. J. 1999. "An explicit, totally analytic approximate solution for Blasius viscous flow problems". *Int. J. Non-Linear Mech.* 34(4): 759–778. DOI: 10.1016/S0020-7462(98)00056-0.

Liao, S. J. 2003. *Beyond Perturbation: Introduction to the Homotopy Analysis Method. Modern Mechanics and Mathematics*. Taylor & Francis.

Liao, S. J. 2013. *Advances in the Homotopy Analysis Method*. World Scientific.

Liao, S. 1992. "The proposed homotopy analysis technique for the solution of nonlinear problems". *PhD thesis*. Shanghai Jiao Tong University.

Lim, Y. H. and D. C. Hamill. 1998. "Nonlinear phenomena in a model spacecraft power system". In: *6th IEEE Workshop on Computers in Power Electronics (COMPEL)*. 169–175.

Litvinov, G. L. 1994. "Approximate construction of rational approximations and the effect of error autocorrection. Applications". *Russian J. Math. Phys.* (in Russian). 1(3): 14–18.

Litvinov, G. L. 2003. "Error autocorrection in rational approximation and interval estimates. [A survey of results.]" *Cent. Eur. J. Math.* 1(1): 36–60. DOI: 10.2478/bf02475663.

Liu, C., B. Wang, F. Hu, K. Sun, and C. L. Bak. 2017. "Online Voltage Stability Assessment for Load Areas Based on the Holomorphic Embedding Method". *arXiv e-prints*. abs/1702.01464. arXiv: 1702.01464 [cs.CE].

Luke, Y. L. 1980. "Computations of coefficients in the polynomials of Padé approximations by solving systems of linear equations". *J. Comput. Appl. Math.* 6(3): 213–218. DOI: 10.1016/0771-050X(80)90028-5.

Luke, Y. L. 1982. "A note on evaluation of coefficients in the polynomials of Padé approximants by solving systems of linear equations". *J. Comput. Appl. Math.* 8(2): 93–99. DOI: 10.1016/0771-050X(82)90061-4.

Mahata, P. and P. S. Nagendra Rao. 2016. "Determination of multiple solutions of load flow equations". *Sādhanā*. 41(8): 855–867. DOI: 10.1007/s12046-016-0524-5.

Marecek, J., T. McCoy, and M. Mevissen. 2014. "Power Flow as an Algebraic System". *arXiv e-prints*. abs/1412.8054. arXiv: 1412.8054 [math.OC].

Marsden, J. and M. Hoffman. 1999. *Basic Complex Analysis*. W. H. Freeman. ISBN: 9780716728771.

McDonald, S. W., C. Grebogi, E. Ott, and J. A. Yorke. 1985. "Fractal basin boundaries". *Phys. D: Nonlinear Phenom.* 17(2): 125–153.

Mehta, D., D. K. Molzahn, and K. Turitsyn. 2016a. "Recent advances in computational methods for the power flow equations". In: *2016 American Control Conference (ACC)*. 1753–1765. DOI: 10.1109/ACC.2016.7525170.

Mehta, D., H. D. Nguyen, and K. Turitsyn. 2016b. "Numerical polynomial homotopy continuation method to locate all the power flow solutions". *IET Gener. Transm. Distribution*. 10(12): 2972–2980. DOI: 10.1049/iet-gtd.2015.1546.

Melville, R. C., L. Trajković, S.-C. Fang, and L. T. Watson. 1993. "Artificial parameter homotopy methods for the DC operating point problem". *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.* 12(6): 861–877.

Metcalf, K. J. 2011. "Power Management and Distribution (PMAD) Model Development: Final Report". *Tech. rep.* No. CR-2011-217268. NASA Glenn Research Center.

Miranda, R. 1995. *Algebraic Curves and Riemann Surfaces*. Vol. 5. *Graduate studies in mathematics*. American Mathematical Society.

Montes, A. 1998. "Algebraic solution of the load-flow problem for a 4-nodes electrical network". *Math. Comput. Simul.* 45(1-2): 163–174. ISSN: 0378-4754. DOI: 10.1016/S0378-4754(97)00092-X.

Montes, A. and J. Castro. 1995. "Solving the Load Flow Problem Using Gröbner Basis". *SIGSAM Bull.* 29(1): 1–13. DOI: 10.1145/216685.216686.

Mori, H. 2000. "Chaotic behavior of the Newton-Raphson method with the optimal multiplier for ill-conditioned power systems". In: *IEEE International Symposium on Circuits and Systems (ISCAS)*. Vol. 4. 237–240. DOI: 10.1109/ISCAS.2000.858732.

Murray, W., T. Tinoco De Rubira, and A. Wigington. 2013. "Improving the robustness of Newton-based power flow methods to cope with poor initial points". In: *North American Power Symposium*. 1–6. DOI: 10.1109/NAPS.2013.6666905.

Murray, W., T. Tinoco De Rubira, and A. Wigington. 2015. "A robust and informative method for solving large-scale power flow problems". *Comput. Optim. Appl.* 62(2): 431–475. DOI: 10.1007/s10589-015-9745-5.

Nguyen, H. D. and K. S. Turitsyn. 2014. "Appearance of multiple stable load flow solutions under power flow reversal conditions". In: *IEEE PES General Meeting*. 1–5. DOI: 10.1109/PESGM.2014.6938797.

Ning, J., W. Gao, G. Radman, and J. Liu. 2009. "The application of the Groebner Basis technique in power flow study". In: *North American Power Symposium*. 1–7. DOI: 10.1109/NAPS.2009.5484033.

Niven, I. 1969. "Formal power series". *Am. Math. Mon.* 76(8): 871–889.

Nuttall, J. 1977. "On convergence of Padé approximants to functions with branch points". In: *Padé and Rational Approximation*. Ed. by E. B. Saff and R. S. Varga. Academic Press, New York. 101–109.

Ortega-Cerdà, J. and B. Pridhnani. 2010. "The Pólya-Tchebotaröv problem". In: *Harmonic Analysis and Partial Differential Equations*. Ed. by P. Cifuentes. Vol. 505. *Contemporary Mathematics*. American Mathematical Society. 153–170. DOI: 10.1090/conm/505.

Paszkowski, S. 1990. "Hermite-Padé approximation (basic notions and theorems)". *J. Comput. Appl. Math.* 32(1–2): 229–236. DOI: 10.1016/0377-0427(90)90433-Z.

Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. 2007. *Numerical Recipes 3rd Edition: The Art of Scientific Computing*. Cambridge University Press.

Quarles, T., D. Pederson, R. Newton, A. Sangiovanni-Vincentelli, and C. Wayne. "SPICE, a general-purpose circuit simulation program". EECS Department of the University of California at Berkeley.

Rakhmanov, E. A. 2016. "The Gonchar-Stahl ρ^2-theorem and associated directions in the theory of rational approximations of analytic functions". *Sb. Math.* 207(9): 1236–1266. DOI: 10.1070/SM8448. URL: http://stacks.iop.org/1064-5616/207/i=9/a=1236.

Rao, S. D. 2017. "Exploration of a Scalable Holomorphic Embedding Method Formulation for Power System Analysis Applications". *PhD thesis*. Arizona State University.

Rao, S. D. and D. J. Tylavsky. 2018. "Theoretical convergence guarantees versus numerical convergence behavior of the holomorphically embedded power flow method". *Int. J. Elec. Power*. 95: 166–176. DOI: 10.1016/j.ijepes.2017.08.018.

Rao, S. D., D. J. Tylavsky, and Y. Feng. 2017a. "Estimating the saddle-node bifurcation point of static power systems using the holomorphic embedding method". *Int. J. Elec. Power*. 84: 1–12. DOI: 10.1016/j.ijepes.2016.04.045.

Rao, S. D., D. J. Tylavsky, W. Yi, V. Vittal, D. Shi, and Z. Wang. 2017b. "Two-Bus Holomorphic Embedding Method-based Equivalents and Weak-Bus Determination". *arXiv e-prints*. abs/1706.01298. arXiv: 1706.01298 [cs.SY].

Rao, S., Y. Feng, D. J. Tylavsky, and M. K. Subramanian. 2016. "The Holomorphic Embedding Method Applied to the Power-Flow Problem". *IEEE Trans. Power Syst.* 31(5): 3816–3828. DOI: 10.1109/TPWRS.2015.2503423.

Roman, C. and W. Rosehart. 2005. "Complementarity model for generator buses in OPF-based maximum loading problems". *IEEE Trans. Power Syst.* 20(1): 514–516. DOI: 10.1109/TPWRS.2004.841152.

Rosehart, W., C. Roman, and A. Schellenberg. 2005. "Optimal power flow with complementarity constraints". *IEEE Trans. Power Syst.* 20(2): 813–822. DOI: 10.1109/TPWRS.2005.846171.

Roychowdhury, J. and R. C. Melville. 2006. "Delivering global DC convergence for large mixed-signal circuits via homotopy/continuation methods". *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.* 25(1): 66–78.

Rutishauser, H. 1957. *Der Quotienten-Differenzen-Algorithmus*. Basel: Birkhäuser. DOI: 10.1007/978-3-0348-7175-4.

Ryder, F. L. 1952. "Network analysis by least power theorems". *J. Frankl. Inst.* 254(1): 47–60. DOI: 10.1016/0016-0032(52)90005-7.

Sarnari, A. J. and R. Živanović. 2016. "Robust Padé approximation for the holomorphic embedding load flow". In: *2016 Australasian Universities Power Engineering Conference (AUPEC)*. 1–6. DOI: 10.1109/AUPEC.2016.7749303.

Sauer, P. W. 1981. "Explicit Load Flow Series and Functions". *IEEE Trans. Power App. Syst.* PAS-100(8): 3754–3763. ISSN: 0018-9510. DOI: 10.1109/TPAS.1981.317018.

Schaffer, M. D. and D. J. Tylavsky. 1988. "A nondiverging polar-form Newton-based power flow". *IEEE Trans. Ind. Appl.* 24(5): 870–877. DOI: 10.1109/28.8993.

Schiefermayr, K. 2014. "The Pólya-Chebotarev problem and inverse polynomial images". *Acta Math. Hung.* 142(1): 80–94. DOI: 10.1007/s10474-013-0353-5.

Sidi, A. 2003. *Practical Extrapolation Methods: Theory and Applications*. Cambridge University Press. ISBN: 9780521661591.

Springer, G. 2001. *Introduction to Riemann Surfaces. AMS Chelsea Publishing Series*. American Mathematical Society.

Stahl, H. 1985a. "Extremal domains associated with an analytic function I, II". *Complex Var. Theory Appl.: Int. J.* 4(4): 311–324, 325–338. DOI: 10.1080/17476938508814117. eprint: http://www.tandfonline.com/doi/pdf/10.1080/17476938508814117.

Stahl, H. 1985b. "The structure of extremal domains associated with an analytic function". *Complex Var. Theory Appl.: Int. J.* 4(4): 339–354. DOI: 10.1080/17476938508814119. eprint: http://www.tandfonline.com/doi/pdf/10.1080/17476938508814119.

Stahl, H. 1989. "On the convergence of generalized Padé approximants". *Constr. Approx.* 5(1): 221–240. DOI: 10.1007/BF01889608.

Stahl, H. 1997. "The Convergence of Padé Approximants to Functions with Branch Points". *J. Approx. Theory.* 91(2): 139–204. DOI: 10.1006/jath.1997.3141.

Stahl, H. 1998. "Spurious poles in Padé approximation". *J. comput. appl. math.* 99(1): 511–527.

Stott, B. 1972. "Decoupled Newton Load Flow". *IEEE Trans. Power App. Syst.* PAS-91(5): 1955–1959. DOI: 10.1109/TPAS.1972.293524.

Stott, B. 1974. "Review of load-flow calculation methods". *Proceedings of the IEEE.* 62(7): 916–929. ISSN: 0018-9219. DOI: 10.1109/PROC.1974.9544.

Stott, B. and O. Alsac. 1974. "Fast Decoupled Load Flow". *IEEE Trans. Power App. Syst.* PAS-93(3): 859–869. DOI: 10.1109/TPAS.1974.293985.

Sturmfels, B. 2002. *Solving systems of polynomial equations. Regional conference series in mathematics.* American Mathematical Society.

Subramanian, M. K., Y. Feng, and D. Tylavsky. 2013. "PV bus modeling in a holomorphically embedded power-flow formulation". In: *North American Power Symposium.* 1–6. DOI: 10.1109/NAPS.2013.6666940.

Tamura, Y., H. Mori, and S. Iwamoto. 1983. "Relationship Between Voltage Instability and Multiple Load Flow Solutions in Electric Power Systems". *IEEE Trans. Power App. Syst.* PAS-102(5): 1115–1125. DOI: 10.1109/TPAS.1983.318052.

Tate, J. E. and T. J. Overbye. 2005. "A comparison of the optimal multiplier in polar and rectangular coordinates". *IEEE Trans. Power Syst.* 20(4): 1667–1674. DOI: 10.1109/TPWRS.2005.857388.

Thorp, J. S. and S. A. Naqavi. 1989. "Load flow fractals". In: *Proceedings of the 28th IEEE Conference on Decision and Control*. Vol. 2. 1822–1827. DOI: 10.1109/CDC.1989.70472.

Thorp, J. S. and S. A. Naqavi. 1997. "Load-flow fractals draw clues to erratic behaviour". *IEEE Comput. Appl. Power*. 10(1): 59–62. DOI: 10.1109/67.560872.

Tinney, W. F. and C. E. Hart. 1967. "Power Flow Solution by Newton's Method". *IEEE Trans. Power App. Syst.* PAS-86(11): 1449–1460. DOI: 10.1109/TPAS.1967.291823.

Tinney, W. F. and J. W. Walker. 1967. "Direct solutions of sparse network equations by optimally ordered triangular factorization". *Proc. IEEE*. 55(11): 1801–1809. DOI: 10.1109/PROC.1967.6011.

Trajković, L. 1999. "Homotopy methods for computing DC operating points". *Wiley Encyclopedia of Electrical and Electronics Engineering*.

Trajković, L. 2012. "DC operating points of transistor circuits". *Nonlinear Theory Appl.* 3(3): 287–300. DOI: 10.1588/nolta.3.287.

Trias, A. 2009. "System and method for monitoring and managing electrical power transmission and distribution networks". *US Patent*. 7519506. URL: http://www.patentlens.net/patentlens/patent/US_7519506.

Trias, A. 2011. "System and method for monitoring and managing electrical power transmission and distribution networks". *US Patent*. 7979239. URL: http://www.patentlens.net/patentlens/patent/US_7979239.

Trias, A. 2012. "The Holomorphic Embedding Load Flow method". In: *Power and Energy Society General Meeting, 2012 IEEE*. 1–8. DOI: 10.1109/PESGM.2012.6344759.

Trias, A. 2015. "Fundamentals of the Holomorphic Embedding Load-Flow Method". *arXiv e-prints*. abs/1509.02421. arXiv: 1509.02421 [cs.SY].

Trias, A. 2017a. "Method to obtain all power flow solutions in radial chain-like networks". *US Patent Application*. Serial No. 62/488,810.

Trias, A. 2017b. "Sigma algebraic approximants as a diagnostic tool in power networks". *US Patent*. 9563722 B2. URL: https://www.lens.org/lens/patent/US_9563722_B2.

Trias, A. and J. L. Marin. 2016. "The Holomorphic Embedding Loadflow Method for DC Power Systems and Nonlinear DC Circuits". *IEEE Trans. Circuits Syst. I*. 63(2): 322–333. DOI: 10.1109/TCSI.2015.2512723.

Trias, A. and J. L. Marin. 2018. "A Padé-Weierstrass technique for the rigorous enforcement of control limits in power flow studies". DOI: 10.1016/j.ijepes.2018.01.033.

Tylavsky, D. J. and M. D. Schaffer. 1987. "A Nondiverging Power Flow Using a Least-Power-Type Theorem". *IEEE Trans. Ind. Appl.* IA-23(5): 944–951. DOI: 10.1109/TIA.1987.4505006.

Vajravelu, K. and R. Van Gorder. 2013. *Nonlinear Flow Phenomena and Homotopy Analysis: Fluid Flow and Heat Transfer*. Springer Berlin Heidelberg.

Van Assche, W. 2006. "Padé and Hermite-Padé approximation and orthogonality". *Surv. Approx. Theory*. 2: 61–91. URL: https://www.emis.de/journals/SAT/papers/6/.

Vlachogiannis, J. G. 1994. "Control adjustments in fast decoupled load flow". *Electr. Power Syst. Res.* 31(3): 185–194. DOI: 10.1016/0378-7796(94)90078-7.

Vournas, C. D., M. Karystianos, and N. G. Maratos. 2000. "Bifurcation points and loadability limits as solutions of constrained optimization problems". In: *2000 Power Engineering Society Summer Meeting*. Vol. 3. 1883–1888 vol. 3. DOI: 10.1109/PESS.2000.868821.

Vu, K. T. and C. C. Liu. 1992. "Shrinking stability regions and voltage collapse in power systems". *IEEE Trans. Circuits Syst. I*. 39(4): 271–289. DOI: 10.1109/81.129455.

Wall, H. S. 1967. *Analytic Theory of Continued Fractions*. Chelsea Pub Co; Reprint edition.

Wallace, I., D. Roberts, A. Grothey, and K. I. M. McKinnon. 2016. "Alternative PV Bus Modelling with the Holomorphic Embedding Load Flow Method". *arXiv e-prints*. abs/1607.00163. arXiv: 1607.00163 [math.OC].

Wang, B., C. Liu, and K. Sun. 2017. "Multi-Stage Holomorphic Embedding Method for Calculating the Power-Voltage Curve". *IEEE Trans. Power Syst. to appear*. DOI: 10.1109/TPWRS.2017.2711260.

Ward, J. B. and H. W. Hale. 1956. “Digital Computer Solution of Power-Flow Problems”. *AIEE Trans. Power App. Syst.* 75(3): 398–404. DOI: 10.1109/AIEEPAS.1956.4499318.

Weber, J. 2004. “Working with Hard-To-Solve Power Flow Cases”. PowerWorld Users Group Meeting. URL: https://www.powerworld.com/knowledge-base/working-with-hard-to-solve-power-flow-cases.

Weniger, E. J. 1989. “Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series”. *Comput. Phys. Rep.* 10(5): 189–371.

Willson, A. N. 1975. *Nonlinear networks: theory and analysis. IEEE Press selected reprint series.* IEEE Press.

Wolf, D. M. and S. R. Sanders. 1996. “Multiparameter homotopy methods for finding DC operating points of nonlinear circuits”. *IEEE Trans. Circuits Syst. I.* 43(10): 824–838.

Wu, F. F. 1977. “Theoretical study of the convergence of the fast decoupled load flow”. *IEEE Trans. Power App. Syst.* 96(1): 268–275. DOI: 10.1109/T-PAS.1977.32334.

Wynn, P. 1956. “On a device for computing the $e_m(S_n)$ transformation”. *Math. Comput.* 10: 91–96.

Wynn, P. 1966. “On the convergence and stability of the epsilon algorithm”. *SIAM J. Numer. Anal.* 3(1): 91–122.

Xu, W., Y. Liu, J. Salmon, T. Le, and G. Chang. 1998. “Series load flow: a novel noniterative load flow method”. *Generation, Transmission and Distribution, IEE Proceedings-*. 145(3): 251–256. ISSN: 1350-2360. DOI: 10.1049/ip-gtd:19981980.

Yue, X. and V. Venkatasubramanian. 2007. “Complementary limit induced bifurcation theorem and analysis of Q limits in power-flow studies”. In: *2007 iREP Symposium - Bulk Power System Dynamics and Control - VII Revitalizing Operational Reliability.* 1–8. DOI: 10.1109/IREP.2007.4410549.

Zaborsky, J. and M. Ilić. 2000. *Dynamics and Control of Large Electric Power Systems.* Wiley-IEEE Press.

Zaki, M. H., I. M. Mitchell, and M. R. Greenstreet. 2009. “DC operating point analysis–A formal approach”. *Proceedings of Formal Verification of Analog Circuits (FAC).*

Zambroni De Souza, A. C., C. B. Rosa, B. I. Lima Lopes, R. C. Leme, and O. A. S. Carpinteiro. 2007. "Non-iterative load-flow method as a tool for voltage stability studies". *IET Gener. Transm. Distribution*. 1(3): 499–505. DOI: 10.1049/iet-gtd:20060310.

Zhao, J., H. D. Chiang, P. Ju, and H. Li. 2008. "On PV–PQ bus type switching logic in power flow computation". In: *Power Systems Computation Conference (PSCC), Glasgow*. http://www.pscc-central.org/uploads/tx_ethpublications/pscc2008_128.pdf. 1–7.

Zhao, X. Y. and X. Zhang. 2008. "Smoothness of voltage stability surface with generator reactive power limit encountered". In: *Electric Utility Deregulation and Restructuring and Power Technologies, 2008. DRPT 2008. Third International Conference on*. 615–619. DOI: 10.1109/DRPT.2008.4523479.

Zhu, Y., D. Tylavsky, and S. Rao. 2017. "Nonlinear structure-preserving network reduction using holomorphic embedding". *IEEE Trans. Power Syst. to appear*. DOI: 10.1109/TPWRS.2017.2738320.

Zimmerman, R. D., C. E. Murillo-Sánchez, and R. J. Thomas. 2011. "MATPOWER: Steady-State Operations, Planning, and Analysis Tools for Power Systems Research and Education". *IEEE Trans. Power Syst.* 26(1): 12–19. ISSN: 0885-8950. DOI: 10.1109/TPWRS.2010.2051168.

Živanovič, R. 2016. "Continuation via quadratic approximation to reconstruct solution branches and locate singularities in the power flow problem". In: *Control and Automation (MED), 24th Mediterranean Conference on*. IEEE. 866–870.