A Friendly Tutorial on Mean-Field Spin Glass Techniques for Non-Physicists

Other titles in Foundations and Trends® in Machine Learning

Dynamical Variational Autoencoders: A Comprehensive Review
Laurent Girin, Simon Leglaive, Xiaoyu Bie, Julien Diard,
Thomas Hueber and Xavier Alameda-Pineda
ISBN: 978-1-68083-912-8

Machine Learning for Automated Theorem Proving: Learning to Solve SAT and QSATe
Sean B. Holden
ISBN: 978-1-68083-898-5

Spectral Methods for Data Science: A Statistical Perspective
Yuxin Chen, Yuejie Chi, Jianqing Fan and Cong Ma
ISBN: 978-1-68083-896-1

Tensor Regression
Jiani Liu, Ce Zhu, Zhen Long and Yipeng Liu
ISBN: 978-1-68083-886-2

Minimum-Distortion Embedding
Akshay Agrawal, Alnur Ali and Stephen Boyd
ISBN: 978-1-68083-888-6

Graph Kernels: State-of-the-Art and Future Challenges
Karsten Borgwardt, Elisabetta Ghisu, Felipe Llinares-López,
Leslie O'Bray and Bastian Rieck
ISBN: 978-1-68083-770-4

A Friendly Tutorial on Mean-Field Spin Glass Techniques for Non-Physicists

Andrea Montanari
Stanford University
montanari@stanford.edu

Subhabrata Sen
Harvard University

the essence of knowledge
Boston — Delft

Foundations and Trends® in Machine Learning

Published, sold and distributed by:
now Publishers Inc.
PO Box 1024
Hanover, MA 02339
United States
Tel. +1-781-985-4510
www.nowpublishers.com
sales@nowpublishers.com

Outside North America:
now Publishers Inc.
PO Box 179
2600 AD Delft
The Netherlands
Tel. +31-6-51115274

The preferred citation for this publication is

A. Montanari and S. Sen. *A Friendly Tutorial on Mean-Field Spin Glass Techniques for Non-Physicists*. Foundations and Trends® in Machine Learning, vol. 17, no. 1, pp. 1–173, 2024.

ISBN: 978-1-63828-213-6

Foundations and Trends® in Machine Learning
Volume 17, Issue 1, 2024

Editorial Scope

Topics

Foundations and Trends® in Machine Learning publishes survey and tutorial articles in the following topics:

- Adaptive control and signal processing
- Applications and case studies
- Behavioral, cognitive and neural learning
- Bayesian learning
- Classification and prediction
- Clustering
- Data mining
- Dimensionality reduction
- Evaluation
- Game theoretic learning
- Graphical models
- Independent component analysis
- Inductive logic programming
- Kernel methods
- Markov chain Monte Carlo
- Model choice
- Nonparametric methods
- Online learning
- Optimization
- Reinforcement learning
- Relational learning
- Robustness
- Spectral methods
- Statistical learning theory
- Variational inference
- Visualization

Information for Librarians

Foundations and Trends® in Machine Learning, 2024, Volume 17, 6 issues. ISSN paper version 1935-8237. ISSN online version 1935-8245. Also available as a combined paper and online subscription.

Contents

A Friendly Tutorial on Mean-Field Spin Glass Techniques for Non-Physicists

Andrea Montanari[1] and Subhabrata Sen[2]

[1] *Department of Electrical Engineering and Department of Statistics, Stanford University, USA; montanari@stanford.edu*
[2] *Department of Statistics, Harvard University, USA*

ABSTRACT

Mean-field spin glasses are a class of high-dimensional random cost (energy) function with special exchangeability properties. Random probability measures are defined from these energy functions by the usual Boltzmann formula. Over the last 40 years, an arsenal of sophisticated mathematical techniques (both rigorous and non-rigorous) has been developed to characterize these models. More recently, these techniques have been successfully applied to a number of canonical models in high-dimensional statistics and machine learning, which fit the same framework. This tutorial provides an introduction to such techniques, aimed at non-specialists.

Andrea Montanari and Subhabrata Sen (2024), "A Friendly Tutorial on Mean-Field Spin Glass Techniques for Non-Physicists", Foundations and Trends® in Machine Learning: Vol. 17, No. 1, pp 1–173. DOI: 10.1561/2200000105.

Introduction

> I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. [...] it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge.
>
> —Archimedes, On the method

This tutorial is based on lecture notes written for a class taught in the Statistics Department at Stanford in the Winter Quarter of 2017 and then again in Fall 2021. The class was called “Methods From Statistical Physics (Stats 369)” and was addressed to students from Statistics, Mathematics, and Engineering Departments with a solid background in probability theory, but no previous knowledge of physics. The objective was to provide a working knowledge of some of the techniques developed over the last 40 years by theoretical physicists and mathematicians to study mean field spin glasses and their applications to high-dimensional statistics and statistical learning.

Why Mean Field Spin Glasses?

The history of the subject in truly remarkable.[1] Spin glass models were introduced by physicists in the 1970s to model the statistical properties of certain magnetic materials. Over the last half century, these models have motivated a blossoming line of mathematical work with applications to multiple fields, at first sight distant from physics.

From a mathematical point of view, spin glasses are high-dimensional probability distributions, i.e., probability distributions over $\mathbb{R}^n$, $n \gg 1$, which are usually written in the Gibbs–Boltzmann form

$$\mu(\mathrm{d}\boldsymbol{\sigma}) \propto e^{\beta H(\boldsymbol{\sigma})}\,\nu_0(\mathrm{d}\boldsymbol{\sigma}). \tag{1}$$

Here ν_0 is a "simple" reference measure (for instance the uniform distribution over $\{+1, -1\}^n$ or the uniform distribution over the sphere S^{n-1}), the exponential weight $H\colon \mathbb{R}^n \to \mathbb{R}$ is known as the Hamiltonian (the standard convention in physics is to refer to $-H$ as the Hamiltonian), and $\beta > 0$ is known as the inverse temperature.

Of course, any probability measure in $\mathbb{R}^n$ can be written in the form (1). However in spin glass models, $H(\boldsymbol{\sigma})$ is typically a sum of polynomially many (in n) "simple" terms (e.g., low degree monomials in the coordinates σ_i), and hence the form (1) is meaningful. It is worth mentioning that, while we focus for simplicity on probability measures over $\mathbb{R}^n$, spin glass models have been studied in other product spaces $\mathcal{X}^n$ as well.

In spin glass models, the Hamiltonian $H(\,\cdot\,)$ itself is random or, to be precise, $\{H(\boldsymbol{\sigma})\}_{\boldsymbol{\sigma}\in\mathbb{R}^n}$ is a stochastic process indexed by $\boldsymbol{\sigma} \in \mathbb{R}^n$ (or $\boldsymbol{\sigma} \in \Sigma$, where Σ denotes the support of ν_0). Therefore the measure $\mu(\mathrm{d}\boldsymbol{\sigma})$ is a random probability measure. A specific spin glass model is defined by specifying the distribution of the process H (alongside the reference measure ν_0).

[1] The reader interested in early historical overviews might wish to consult the sequence of seven expository articles written by Phil Anderson, between 1988 (Anderson, 1988) and 1990 (Anderson, 1990), or of course the unsurpassed collection of articles (and accompanying introductions) in Mézard *et al.* (1987). For a personal account, see Charbonneau (2021).

At first sight, studying random probability measures might seem a somewhat exotic endeavor. However, a little thought reveals that random probability measures are both ubiquitous and useful:

(1) Consider a statistical estimation problem: we want to estimate an unknown vector $\boldsymbol{x} \in \mathbb{R}^n$ from some data $\boldsymbol{Y}$. We will see several examples of this problem in this tutorial. A Bayesian approach postulates a prior distribution over $\boldsymbol{x}$, and then forms a posterior, namely the conditional distribution of $\boldsymbol{x}$ given $\boldsymbol{Y}$, under that prior. The posterior is a random probability distribution over $\mathbb{R}^n$ (because $\boldsymbol{Y}$ is random).

(2) Consider an optimization problem of the form $\max_{\boldsymbol{\sigma}\in\Sigma} H(\boldsymbol{\sigma})$. In many circumstances, we have a probabilistic model for H. For instance, this is the case in empirical risk minimization, which is the standard approach to statistical learning. Of course, in order to understand the properties of this optimization problem, it is important to understand the geometry of the (random) superlevel sets $\Sigma_E := \{\boldsymbol{\sigma} \in \Sigma\colon H(\boldsymbol{\sigma}) \geq E\}$. It turns out that—in many cases—the distribution (1) is closely related to the uniform distribution over $\Sigma_{E(\beta)}$ for a certain $E(\beta)$. Therefore, the Gibbs measure (1) is a powerful tool to explore the random geometry of superlevel sets.

(3) In physics, the Hamiltonian H is random because, for instance, the spins σ_i are magnetic moments associated to impurities at random locations in an otherwise non-magnetic material. More generally, H can be random because it is the Hamiltonian of a disordered system, whose "disorder" degrees of freedom are not thermalized.

Mean field spin glass models are a special subfamily of spin glass models. Roughly speaking, they are characterized by the fact that the coordinates of $\boldsymbol{\sigma}$ (the "spins" in physics language) are "indistinguishable" from the point of view of the process[2] $H(\,\cdot\,)$. The first model

[2]Formally, for any fixed number of vectors $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k$, the joint distribution of $(H(\boldsymbol{\sigma}_1), \ldots, H(\boldsymbol{\sigma}_k))$ depends on $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k$ only through the joint empirical distribution of their coordinates $n^{-1}\sum_{i=1}^n \delta_{\sigma_{1,i},\ldots,\sigma_{k,i}}$.

of this type, which we will study in Section 2, was first introduced by David Sherrington and Scott Kirkpatrick in 1975 (Sherrington and Kirkpatrick, 1975) to provide an idealized model that could be amenable to mathematical analysis.

In the near half-century since then, and starting with the invention of "replica symmetry breaking" by Parisi (1979a), physicists have developed a number of sophisticated non-rigorous techniques to analyze mean field spin glasses and characterize their high-dimensional behavior. While several of the physicists' results and technique are outstanding challenges for mathematicians, since the early 2000s, there has been increasing success in rigorizing some of these ideas. These developments have given birth to a rich and rapidly evolving area of probability theory.

In parallel with these developments, it has become increasingly clear that understanding the behavior of random high-dimensional probability distributions $\mu(\mathrm{d}\boldsymbol{\sigma})$ and of random high-dimensional objective functions $H(\boldsymbol{\sigma})$ is crucial in a number of mathematical disciplines beyond theoretical physics. We mentioned above high-dimensional statistics and statistical learning: tools and intuitions from physics have found countless applications in these areas in the last 10–15 years. In the opposite direction, high-dimensional statistics and statistical learning have brought new questions and stimulated new developments in spin glass theory.

This tutorial is mainly aimed at researchers in statistics, mathematics, computer science, who want to learn some of the important tools and ideas in this area.

The Style of This Tutorial

This tutorial is deliberately written in a somewhat non-standard style, from several viewpoints:

Concrete problems. Rather than developing the theory in the most general setting, we focus on two concrete problems that are motivated by questions in statistical estimation. Each of the next two sections is dedicated to one such problem.

We use each of these examples as a pretext for presenting a number of mathematical techniques. We believe it is best to learn these technique on concrete applications.

Non-exhaustive. Our treatment is far from exhaustive, even for each of the specific problems that we treat. On the other hand, while we use these examples as motivation, we do not hesitate in pursuing detours that are interesting, but indirectly related to the original questions posed by those examples.

Rigorous vs. non-rigorous techniques. We present a mixture of non-rigorous and rigorous techniques. Whenever something is proven (or a proof in the literature is indicated, or sketched) we emphasize this by using the labels "theorem," "lemma," and so on. On the other hand, we explain non-rigorous techniques on examples for which rigorous alternatives (yielding the same conclusions) are available.

There are countless reasons for learning non-rigorous techniques in parallel with rigorous ones. Among others: (i) They have driven this research area; (ii) Properly used, they provide the correct answer with significantly less work; (iii) They apply more broadly; (iv) They can provide invaluable insights/conjectures for rigorous research.

As shown by the quote above, reason (iv) was already acknowledged by Archimedes more than two millennia ago.

Exercises. As explained above, this tutorial is based on a class taught at Stanford. We include the exercises developed for that class (often generalizations of the models treated in the main text). The importance of hands-on practice in mathematics cannot be overstated. Even more so when learning a completely different point of view (e.g., learning non-rigorous techniques for a mathematically minded person, or vice versa).

References

Adler, R. J. and J. E. Taylor (2007). *Random Fields and Geometry*. Vol. 80. New York: Springer.

Aizenman, M., R. Sims, and S. L. Starr (2003). "Extended variational principle for the Sherrington–Kirkpatrick spin-glass model". *Physical Review B*. 68(21): 214403. DOI: 10.1103/PhysRevB.68.214403.

Anderson, P. W. (1988). "Spin glass I: A scaling law rescued". *Physics Today*. 41(1): 9–11.

Anderson, P. W. (1990). "Spin glass VII: Spin glass as a paradigm". *Physics Today*. 43(3): 9–11. DOI: 10.1063/1.2810479.

Arora, S., E. Berger, H. Elad, G. Kindler, and M. Safra (2005). "On non-approximability for quadratic programs". In: *46th Annual IEEE Symposium on Foundations of Computer Science, 2005. FOCS 2005.* IEEE. 206–215.

Arous, G. B. and A. Guionnet (1997). "Large deviations for Wigner's law and Voiculescu's non-commutative entropy". *Probability Theory and Related Fields*. 108(4): 517–542. DOI: 10.1007/s004400050119.

Auffinger, A., G. B. Arous, and J. Černy (2013). "Random matrices and complexity of spin glasses". *Communications on Pure and Applied Mathematics*. 66(2): 165–201. DOI: 10.1002/cpa.21422.

Auffinger, A. and W.-K. Chen (2015a). "On properties of Parisi measures". *Probability Theory and Related Fields*. 161(3–4): 817–850. DOI: 10.1007/s00440-014-0563-y.

Auffinger, A. and W.-K. Chen (2015b). "The Parisi formula has a unique minimizer". *Communications in Mathematical Physics*. 335(3): 1429–1444. DOI: 10.1007/s00220-014-2254-z.

Baik, J., G. Ben Arous, and S. Péché (2005). "Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices". *Annals of Probability*. 33(5): 1643–1697. DOI: 10.1214/009117905000000233.

Barak, B. and A. Moitra (2016). "Noisy tensor completion via the sum-of-squares hierarchy". In: *Conference on Learning Theory*. PMLR. 417–445.

Barbier, J., M. Dia, N. Macris, F. Krzakala, and L. Zdeborová (2018). "Rank-one matrix estimation: Analysis of algorithmic and information theoretic limits by the spatial coupling method". *arXiv:1812.02537*.

Barbier, J. and N. Macris (2019). "The adaptive interpolation method: A simple scheme to prove replica formulas in Bayesian inference". *Probability Theory and Related Fields*. 174(3–4): 1133–1185. DOI: 10.1007/s00440-018-0879-0.

Barra, A., P. Contucci, E. Mingione, and D. Tantari (2015). "Multi-species mean field spin glasses. Rigorous results". *Annales Henri Poincaré*. 16(3): 691–708. DOI: 10.1007/s00023-014-0341-5.

Bates, E. and Y. Sohn (2021). "Free energy in multi-species mixed p-spin spherical models". *arXiv preprint arXiv:2109.14790*. DOI: 10.1214/22-EJP780.

Bayati, M. and A. Montanari (2011). "The dynamics of message passing on dense graphs, with applications to compressed sensing". *IEEE Trans. on Inform. Theory*. 57: 764–785. DOI: 10.1109/TIT.2010.2094817.

Bayati, M. and A. Montanari (2012). "The LASSO risk for Gaussian matrices". *IEEE Trans. on Inform. Theory*. 58(4): 1997–2017. DOI: 10.1109/TIT.2011.2174612.

Berthier, R., A. Montanari, and P.-M. Nguyen (2020). "State evolution for approximate message passing with non-separable functions". *Information and Inference: A Journal of the IMA*. 9(1): 33–79. DOI: 10.1093/imaiai/iay021.

Bolthausen, E. (2014). "An iterative construction of solutions of the TAP equations for the Sherrington–Kirkpatrick model". *Communications in Mathematical Physics*. 325(1): 333–366. DOI: 10.1007/s00220-013-1862-3.

Boucheron, S., G. Lugosi, and P. Massart (2013). *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford: Oxford University Press. DOI: 10.1093/acprof:oso/9780199535255.001.0001.

Carmona, P. and Y. Hu (2006). "Universality in Sherrington–Kirkpatrick's spin glass model". *Annales de l'Institut Henri Poincare (B) Probability and Statistics*. 42(2): 215–222. DOI: 10.1016/j.anihpb.2005.04.001.

Celentano, M., A. Montanari, and Y. Wu (2020). "The estimation error of general first order methods". In: *Conference on Learning Theory*. PMLR. 1078–1141.

Charbonneau, P. (2021). *History of RSB Interview: Miguel Virasoro*, transcript of an oral history conducted 2021 by Patrick Charbonneau and Francesco Zamponi, History of RSB Project, CAPHES École normale supérieure, Paris. CAPHES École normale supérieure, Paris. DOI: 10.34847/nkl.a941vym8.

Chatterjee, S. (2005). "A simple invariance theorem". *arXiv preprint math/0508213*.

Chen, W.-K. (2013). "The Aizenman–Sims–Starr scheme and Parisi formula for mixed p-spin spherical models". *Electronic Journal of Probability*. 18: 1–14. DOI: 10.1214/EJP.v18-2580.

Chen, W.-K. (2019). "Phase transition in the spiked random tensor with Rademacher prior". *The Annals of Statistics*. 47(5): 2734–2756. DOI: 10.1214/18-AOS1763.

Chen, W.-K. and W.-K. Lam (2021). "Universality of approximate message passing algorithms". *Electronic Journal of Probability*. 26: 1–44. DOI: 10.1214/21-EJP604.

Cover, T. M. and J. A. Thomas (1991). *Elements of Information Theory*.

Crisanti, A., H. Horner, and H.-J. Sommers (1993). "The spherical p-spin interaction spin-glass model". *Zeitschrift für Physik B Condensed Matter*. 92(2): 257–271. DOI: 10.1007/BF01312184.

Crisanti, A. and T. Rizzo (2002). "Analysis of the ∞-replica symmetry breaking solution of the Sherrington–Kirkpatrick model". *Physical Review E*. 65(4): 046137. DOI: 10.1103/PhysRevE.65.046137.

Crisanti, A. and H.-J. Sommers (1992). "The spherical p-spin interaction spin glass model: The statics". *Zeitschrift für Physik B Condensed Matter*. 87(3): 341–354. DOI: 10.1007/BF01309287.

Crisanti, A. and H.-J. Sommers (1995). "Thouless–Anderson–Palmer approach to the spherical p-spin spin glass model". *Journal de Physique I*. 5(7): 805–813. DOI: 10.1051/jp1:1995164.

Daley, D. J. and D. Vere-Jones (1998). "Introduction to the general theory of point processes". In: *An Introduction to the Theory of Point Processes*. New York: Springer. 197–233. DOI: 10.1007/978-1-4757-2001-3_7.

De Almeida, J. R. L. and E. J. S. Lage (1983). "Internal field distribution in the infinite-range Ising spin glass". *Journal of Physics C: Solid State Physics*. 16(5): 939. DOI: 10.1088/0022-3719/16/5/020.

Dembo, A., A. Montanari, and S. Sen (2017). "Extremal cuts of sparse random graphs". *The Annals of Probability*. 45(2): 1190–1217. DOI: 10.1214/15-AOP1084.

Deshpande, Y., E. Abbe, and A. Montanari (2016). "Asymptotic mutual information for the balanced binary stochastic block model". *Information and Inference: A Journal of the IMA*. 6(2): 125–170. DOI: 10.1093/imaiai/iaw017.

Deshpande, Y., E. Abbe, and A. Montanari (2017). "Asymptotic mutual information for the balanced binary stochastic block model". *Information and Inference: A Journal of the IMA*. 6(2): 125–170. DOI: 10.1093/imaiai/iaw017.

Donoho, D. L., A. Maleki, and A. Montanari (2009). "Message passing algorithms for compressed sensing". *Proceedings of the National Academy of Sciences*. 106: 18914–18919. DOI: 10.1073/pnas.0909892106.

Donoho, D. and A. Montanari (2016). "High dimensional robust m-estimation: Asymptotic variance via approximate message passing". *Probability Theory and Related Fields*. 166(3–4): 935–969. DOI: 10.1007/s00440-015-0675-z.

Edwards, S. F. and P. W. Anderson (1975). "Theory of spin glasses". *Journal of Physics F: Metal Physics*. 5(5): 965. DOI: 10.1088/0305-4608/5/5/017.

El Alaoui, A. and F. Krzakala (2018). "Estimation in the spiked Wigner model: A short proof of the replica formula". In: *2018 IEEE International Symposium on Information Theory (ISIT)*. IEEE. 1874–1878. DOI: 10.1109/ISIT.2018.8437810.

El Alaoui, A., F. Krzakala, and M. Jordan (2020). "Fundamental limits of detection in the spiked wigner model". *The Annals of Statistics*. 48(2): 863–885. DOI: 10.1214/19-AOS1826.

Fan, Z. (2022). "Approximate message passing algorithms for rotationally invariant matrices". *The Annals of Statistics*. 50(1): 197–224.

Fekete, M. (1923). "Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten". *Mathematische Zeitschrift*. 17(1): 228–249. DOI: 10.1007/BF01504345.

Feng, O. Y., R. Venkataramanan, C. Rush, and R. J. Samworth (2022). "A unifying tutorial on approximate message passing". *Foundations and Trends® in Machine Learning*. 15(4): 335–536. DOI: 10.1561/2200000092.

Gallager, R. (1962). "Low-density parity-check codes". *IRE Transactions on Information Theory*. 8(1): 21–28. DOI: 10.1109/TIT.1962.1057683.

Gillin, P. and D. Sherrington (2000). "$p > 2$ spin glasses with first-order ferromagnetic transitions". *Journal of Physics A: Mathematical and General*. 33(16): 3081. DOI: 10.1088/0305-4470/33/16/302.

Guerra, F. (2003). "Broken replica symmetry bounds in the mean field spin glass model". *Communications in Mathematical Physics*. 233(1): 1–12. DOI: 10.1007/s00220-002-0773-5.

Guerra, F. and F. L. Toninelli (2002). "The thermodynamic limit in mean field spin glass models". *Communications in Mathematical Physics*. 230(1): 71–79. DOI: 10.1007/s00220-002-0699-y.

Hoyle, D. C. and M. Rattray (2003). "PCA learning for sparse high-dimensional data". *Europhysics Letters*. 62(1): 117. DOI: 10.1209/epl/i2003-00370-1.

Hoyle, D. C. and M. Rattray (2004). "Principal-component-analysis eigenvalue spectra from data with symmetry-breaking structure". *Physical Review E*. 69(2): 026124. DOI: 10.1103/PhysRevE.69.026124.

Hsu, D., S. M. Kakade, and T. Zhang (2012). "A spectral algorithm for learning hidden Markov models". *Journal of Computer and System Sciences*. 78(5): 1460–1480. DOI: 10.1016/j.jcss.2011.12.025.

Jagannath, A. and S. Sen (2020). "On the unbalanced cut problem and the generalized Sherrington–Kirkpatrick model". *Annales de l'Institut Henri Poincaré D*. 8(1): 35–88. DOI: 10.4171/AIHPD/97.

Jagannath, A. and I. Tobasco (2016). "A dynamic programming approach to the Parisi functional". *Proceedings of the American Mathematical Society*. 144(7): 3135–3150. DOI: 10.1090/proc/12968.

Javanmard, A. and A. Montanari (2013). "State evolution for general approximate message passing algorithms, with applications to spatial coupling". *Information and Inference: A Journal of the IMA*. 2(2): 115–144. DOI: 10.1093/imaiai/iat004.

Javanmard, A., A. Montanari, and F. Ricci-Tersenghi (2016). "Phase transitions in semidefinite relaxations". *Proceedings of the National Academy of Sciences*. 113(16): E2218–E2223. DOI: 10.1073/pnas.1523097113.

Korada, S. B. and N. Macris (2009). "Exact solution of the gauge symmetric p-spin glass model on a complete graph". *Journal of Statistical Physics*. 136(2): 205–230. DOI: 10.1007/s10955-009-9781-6.

Kosterlitz, J. M., D. J. Thouless, and R. C. Jones (1976). "Spherical model of a spin-glass". *Physical Review Letters*. 36(20): 1217. DOI: 10.1103/PhysRevLett.36.1217.

Kreimer, N., A. Stanton, and M. D. Sacchi (2013). "Tensor completion based on nuclear norm minimization for 5D seismic data reconstruction". *Geophysics*. 78(6): V273–V284. DOI: 10.1190/geo2013-0022.1.

Lehmann, E. L. and G. Casella (2006). *Theory of Point Estimation*. New York–Berlin–Heidelberg: Springer Science & Business Media.

Lelarge, M. and L. Miolane (2019). "Fundamental limits of symmetric low-rank matrix estimation". *Probability Theory and Related Fields*. 173(3-4): 859–929. DOI: 10.1007/s00440-018-0845-x.

Lesieur, T., L. Miolane, M. Lelarge, F. Krzakala, and L. Zdeborová (2017). "Statistical and computational phase transitions in spiked tensor estimation". In: *2017 IEEE International Symposium on Information Theory (ISIT)*. IEEE. 511–515. DOI: 10.1109/ISIT.2017.8006580.

Li, N. and B. Li (2010). "Tensor completion for on-board compression of hyperspectral images". In: *2010 IEEE International Conference on Image Processing*. IEEE. 517–520. DOI: 10.1109/ICIP.2010.5651225.

Liggett, T. M. (1978). "Random invariant measures for Markov chains, and independent particle systems". *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*. 45(4): 297–313. DOI: 10.1007/BF00537539.

Liu, J., P. Musialski, P. Wonka, and J. Ye (2013). "Tensor completion for estimating missing values in visual data". *IEEE Transactions on Pattern Analysis and Machine Intelligence*. 35(1): 208–220. DOI: 10.1109/TPAMI.2012.39.

Mézard, M. and A. Montanari (2009). *Information, Physics and Computation*. Oxford: Oxford University Press.

Mézard, M., G. Parisi, and M. A. Virasoro (1986). "Sk model: The replica solution without replicas". *Europhysics Letters*. 1(2): 77.

Mézard, M., G. Parisi, and M. A. Virasoro (1987). *Spin Glass Theory and Beyond*. Singapore: World Scientific. DOI: 10.1142/0271.

Montanari, A., D. Reichman, and O. Zeitouni (2016). "On the limitation of spectral methods: From the Gaussian hidden clique problem to rank one perturbations of Gaussian tensors". *IEEE Transactions on Information Theory*. 63(3): 1572–1579. DOI: 10.1109/TIT.2016.2637959.

Montanari, A. and E. Richard (2014). "A statistical model for tensor PCA". *Advances in Neural Information Processing Systems*. 27. Ed. by Z. Ghahramani, M. Welling, C. Cortes, N. Lawrence, and K. Q. Weinberger. URL: https://proceedings.neurips.cc/paper_files/paper/2014/file/b5488aeff42889188d03c9895255cecc-Paper.pdf.

Montanari, A. and R. Venkataramanan (2021). "Estimation of low-rank matrices via approximate message passing". *The Annals of Statistics*. 49(1): 321–345. DOI: 10.1214/20-AOS1958.

Mørup, M. (2011). “Applications of tensor (multiway array) factorizations and decompositions in data mining”. *Wiley Interdisciplinary Reviews: Data Mining and Knowledge Discovery*. 1(1): 24–40. DOI: 10.1002/widm.1.

Özyec sil, O., V. Voroninski, R. Basri, and A. Singer (2017). “A survey of structure from motion”. *Acta Numerica*. 26: 305–364. DOI: 10.1017/S096249291700006X.

Panchenko, D. (2013). *The Sherrington–Kirkpatrick Model*. New York–Heidelberg–Dordrecht–London: Springer Science & Business Media. DOI: 10.1007/978-1-4614-6289-7.

Panchenko, D. (2014). “The Parisi formula for mixed p-spin models”. *The Annals of Probability*. 42(3): 946–958. DOI: 10.1214/12-AOP800.

Panchenko, D. (2015). “The free energy in a multi-species Sherrington–Kirkpatrick model”. *The Annals of Probability*. 43(6): 3494–3513. DOI: 10.1214/14-AOP967.

Panchenko, D. (2018a). “Free energy in the mixed p-spin models with vector spins”. *The Annals of Probability*. 46(2): 865–896. DOI: 10.1214/17-AOP1194.

Panchenko, D. (2018b). “Free energy in the Potts spin glass”. *The Annals of Probability*. 46(2): 829–864. DOI: 10.1214/17-AOP1193.

Panchenko, D. (2018c). “On the K-sat model with large number of clauses”. *Random Structures & Algorithms*. 52(3): 536–542. DOI: 10.1002/rsa.20748.

Papadimitriou, C. H. and M. Yannakakis (1991). “Optimization, approximation, and complexity classes”. *Journal of Computer and System Sciences*. 43(3): 425–440. DOI: 10.1016/0022-0000(91)90023-X.

Parisi, G. (1979a). “Infinite number of order parameters for spin-glasses”. *Physical Review Letters*. 43(23): 1754. DOI: 10.1103/PhysRevLett.43.1754.

Parisi, G. (1979b). “Toward a mean field theory for spin glasses”. *Physics Letters A*. 73(3): 203–205. DOI: 10.1016/0375-9601(79)90708-4.

Perry, A., A. S. Wein, and A. S. Bandeira (2020). “Statistical limits of spiked tensor models”. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*. 56(1): 230–264. DOI: 10.1214/19-AIHP960.

Rangan, S. (2011). "Generalized approximate message passing for estimation with random linear mixing". In: *IEEE Intl. Symp. on Inform. Theory*. St. Perersbourg. DOI: 10.1109/ISIT.2011.6033942.

Rangan, S., P. Schniter, and A. K. Fletcher (2019). "Vector approximate message passing". *IEEE Transactions on Information Theory*. 65(10): 6664–6684. DOI: 10.1109/TIT.2019.2916359.

Richardson, T. J. and R. Urbanke (2008). *Modern Coding Theory*. Cambridge: Cambridge University Press. DOI: 10.1017/CBO9780511791338.

Ruzmaikina, A. and M. Aizenman (2005). "Characterization of invariant measures at the leading edge for competing particle systems". *The Annals of Probability*. 33(1): 82–113. DOI: 10.1214/009117904000000865.

Schmidt, M. J. (2008). "Replica symmetry breaking at low temperatures". Ph.D. Thesis. URL: https://d-nb.info/991972910/34.

Sen, S. (2018). "Optimization on sparse random hypergraphs and spin glasses". *Random Structures & Algorithms*. 53(3): 504–536. DOI: 10.1002/rsa.20774.

Sherrington, D. and S. Kirkpatrick (1975). "Solvable model of a spin-glass". *Physical Review Letters*. 35(26): 1792. DOI: 10.1103/PhysRevLett.35.1792.

Signoretto, M., R. Van de Plas, B. De Moor, and J. A. K. Suykens (2011). "Tensor versus matrix completion: A comparison with application to spectral data". *IEEE Signal Processing Letters*. 18(7): 403–406. DOI: 10.1109/LSP.2011.2151856.

Subag, E. (2017). "The complexity of spherical p-spin models? A second moment approach". *The Annals of Probability*. 45(5): 3385–3450. DOI: 10.1214/16-AOP1139.

Sur, P. and E. J. Candès (2019). "A modern maximum-likelihood theory for high-dimensional logistic regression". *Proceedings of the National Academy of Sciences*. 116(29): 14516–14525. DOI: 10.1073/pnas.1810420116.

Talagrand, M. (2006a). "Free energy of the spherical mean field model". *Probability Theory and Related Fields*. 134(3): 339–382. DOI: 10.1007/s00440-005-0433-8.

Talagrand, M. (2006b). "The Parisi formula". *Annals of Mathematics*. 163(1): 221–263. DOI: 10.4007/annals.2006.163.221.

Talagrand, M. (2010). *Mean Field Models for Spin Glasses: Volume I: Basic Examples*. Vol. 54. Heidelberg–Dordrecht–London–New York: Springer Science & Business Media. DOI: 10.1007/978-3-642-15202-3.

Talagrand, M. (2011). *Mean Field Models for Spin Glasses: Volume II: Advanced Replica-Symmetry and Low Temperature*. Vol. 55. Heidelberg–Dordrecht–London–New York: Springer Science & Business Media. DOI: 10.1007/978-3-642-22253-5.

Vershynin, R. (2012). "Introduction to the non-asymptotic analysis of random matrices". In: *Compressed Sensing: Theory and Applications*. Ed. by Y.C. Eldar and G. Kutyniok. Cambridge: Cambridge University Press. 210–268. DOI: 10.1017/CBO9780511794308.006.

Wainwright, M. J. and M. I. Jordan (2008). "Graphical models, exponential families, and variational inference". *Foundations and Trends in Machine Learning*. 1(1-2): 1–305. DOI: 10.1561/2200000001.

Zdeborová, L. and S. Boettcher (2010). "A conjecture on the maximum cut and bisection width in random regular graphs". *Journal of Statistical Mechanics: Theory and Experiment*. 2010(02): P02020. DOI: 10.1088/1742-5468/2010/02/P02020.