Performance Bounds and Suboptimal Policies for Multi-Period Investment

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Contents

Abstract

We consider dynamic trading of a portfolio of assets in discrete periods over a finite time horizon, with arbitrary time-varying distribution of asset returns. The goal is to maximize the total expected revenue from the portfolio, while respecting constraints on the portfolio such as a required terminal portfolio and leverage and risk limits. The revenue takes into account the gross cash generated in trades, transaction costs, and costs associated with the positions, such as fees for holding short positions. Our model has the form of a stochastic control problem with linear dynamics and convex cost function and constraints. While this problem can be tractably solved in several special cases, such as when all costs are convex quadratic, or when there are no transaction costs, our focus is on the more general case, with nonquadratic cost terms and transaction costs.

We show how to use linear matrix inequality techniques and semidefinite programming to produce a quadratic bound on the value function, which in turn gives a bound on the optimal performance. This performance bound can be used to judge the performance obtained by any suboptimal policy. As a by-product of the performance bound computation, we obtain an approximate dynamic programming policy that requires the solution of a convex optimization problem, often a quadratic program, to determine the trades to carry out in each step. While we have no theoretical guarantee that the performance of our suboptimal policy is always near the performance bound (which would imply that it is nearly optimal) we observe that in numerical examples the two values are typically close.

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1 Introduction

1.1 Overview

In this paper we formulate the discrete-time finite horizon time-varying multi-period investment problem as a stochastic control problem. By using state variables that track the *value* of the assets, instead of more traditional choices of states such as the number of shares or the fraction of total value, the stochastic control problem has *linear* (but random) dynamics. Assuming that the costs and constraints are convex, we arrive at a linear convex stochastic control problem.

This problem can be effectively solved in two broad cases. When there are no transaction costs, the multi-period investment problem can be reduced to solving a set of standard single-period investment problems; the optimal policy in this case is to simply rebalance the portfolio to a pre-computed optimal portfolio in each step. Another case in which the problem can be effectively solved is when the costs are quadratic and the only constraints are linear equality constraints. In this case standard dynamic programming (DP) techniques can be used to compute the optimal trading policies, which are affine functions of the current portfolio. We describe these special cases in more detail in §3.3 and §3.2. The problem is also tractable when the number of assets is very small, say two or three, in which case brute force (numerical) dynamic programming can be used to compute an optimal policy.

Most problems of interest, however, include significant transaction costs, or include terms that are not well approximated by quadratic functions. In these cases, the optimal investment policy cannot be tractably computed. In such situations, several approaches can be used to find suboptimal policies, including approximate dynamic programming (ADP) and model predictive control (MPC). The performance of any suboptimal policy can be evaluated using Monte Carlo analysis, by simulation over many return trajectories. An obvious practical (and theoretical) question is, how suboptimal is the policy? In this paper we address this question.

Using linear matrix inequality (LMI) techniques widely used in control system analysis and design [18, 35, 80], we construct a (numerical) bound on the best performance that can be attained, for a given problem. The method requires the construction and solution of a semidefinite program (SDP), a convex optimization problem involving matrix inequalities. We can compare the bound on performance with the performance attained by any suboptimal policy; when they are close, we conclude that the policy is approximately optimal (and that the performance bound is nearly tight). Even when the performance bound and suboptimal policy performance are not close, we at least have a bound on how suboptimal our suboptimal policy can be.

The performance bound computation yields a quadratic approximation (in fact, underestimator) of the value functions for the stochastic control problem. These quadratic value function approximations can be used in an ADP policy, or as the terminal cost in an MPC policy. While we have no a priori guarantee that the gap between the performance bound and the performance of the ADP policy will always be small, simulations show that the ADP and MPC policies achieve performance that is often nearly optimal.

Our methods for computing the performance bound, as well as implementing the ADP and MPC suboptimal policies, rely on (numerically) solving convex optimization problems, for which there are efficient and reliable algorithms available [20, 72, 77, 73, 102]. The performance bound computation requires solving SDPs [20, 95], which can be done using modern interior-point cone solvers such as SeDuMi or SDPT3 [88, 92, 94]. Parser-solvers such as CVX or YALMIP [40, 58] allow the user to specify the SDPs in a natural high-level mathematical description form, greatly reducing the time required to form and solve the SDPs. The SDPs that we solve involve T matrices of size $n \times n$, where n is the number of assets, and T is the trading period horizon. These SDPs can be challenging to solve (depending on n and T, of course), using generic methods; but this computation is done once, off-line, before trading begins.

Evaluating the ADP suboptimal policy in each period (*i.e.*, determining the trades to execute) requires solving a small and structured convex optimization problem with (on the order of) n scalar variables. Solving these problems using generic solvers might take seconds, or even minutes, depending on the problem size and types of constraints and objective terms. But recent advances have shown that if the solver is customized for the particular problem family, orders of magnitude speed up is possible [64, 65, 63, 62, 99]. This means that the ADP trading policies we design can be executed at time scales measured in milliseconds or microseconds for modest size problems (say, tens of assets), even with complex constraints. In addition, the trading policies we design can be tested and verified via Monte Carlo simulation very efficiently. For example, the simulation of the numerical examples of the ADP policies reported in this paper required the solution of around 50 million quadratic programs (QPs). These were solved in a few hours on a desktop computer using custom solvers generated by CVXGEN, a code generator for embedded convex optimization [64].

Evaluating the MPC policy also requires the solution of a structured convex optimization problem, with (on the order of) nT variables. If a custom solver is used, the computational effort required is approximately T times the effort required to evaluate the ADP policy. One major advantage of MPC is that it does not require any pre-computation; to implement the ADP policy, we must first solve a large SDP to find the approximate value functions. MPC can thus directly incorporate real-time signals such as changes in future return statistics.

1.2 Prior and related work

Portfolio optimization has been studied and used for more than 60 years. In this section our goal is to give a brief overview of some of the important research in this area, focussing on work related to our approach. Readers interested in a broader overview of the applications of stochastic control and optimization to economics and finance should refer to, *e.g.*, [1, 34, 45, 76, 90, 104].

Single-period portfolio optimization

Portfolio optimization was introduced by Markowitz in 1952 [61]. He formulated a single period portfolio investment problem as a quadratic optimization problem with an objective that trades off expected return and variance. Since this first work, many papers have extended the single period portfolio optimization framework. For example, Goldsmith [38] is one of the first papers to include an analysis of the effect of transaction costs on portfolio selection. Modern convex optimization methods, such as second-order cone programming (SOCP), are applied to portfolio problems with transaction costs in [57, 56]. Convex optimization methods have also been used to handle more sophisticated measures of risk, such as conditional value at risk (CVaR) [82, 50].

Dynamic multi-period portfolio optimization

Early attempts to extend the return-variance trade-off to multi-period portfolio optimization include [91, 71]. One of the first works on multi-period portfolio investment in a dynamic programming framework is by Merton [68]. In this seminal paper, the author considers a problem with one risky asset and one risk-free asset; at each continuous time instant, the investor chooses what proportion of his wealth to invest and what to consume, seeking to maximize the total utility of the wealth consumed over a finite time horizon. When there are no constraints or transaction costs, and under some additional assumptions on the investor utility function, Merton derived a simple closed-form expression for the optimal policy. In a companion paper [84], Samuelson derived the discrete-time analog of Merton's approach.

Constantinides [26] extended Samuelson's discrete-time formulation to problems with proportional transaction costs. In his paper, Constantinides demonstrated the presence of a convex 'no-trade cone'. When the portfolio is within the cone the optimal policy is not to trade; outside the cone, the optimal policy is to trade to the boundary of the cone. (We will see that the policies we derive in this paper have similar properties.) Davis and Norman [29] and Dumas and Lucian [33] derived similar results for the continuous-time formulation. In [28], the authors consider a specific multi-period portfolio problem in continuous time, where they derive a formula for the minimum wealth needed to hedge an arbitrary contingent claim with proportional transaction costs. More recent work includes [93, 23, 24]; in these the authors develop affine recourse policies for discrete time portfolio optimization.

Log-optimal investment

A different formulation for the multi-period problem was developed by Kelly [49], where it was shown that a log-optimal investment strategy maximizes the long-term growth rate of cumulative wealth in horse-race markets. This was extended in [21] to general asset returns and further extended to include all frictionless stationary ergodic markets in [3] and [27]. More recently, Iyengar [44] extended these problems to include proportional transaction costs.

Linear-quadratic multi-period portfolio optimization

Optimal policies for unconstrained linear-quadratic portfolio problems have been derived for continuous-time formulations by Zhou and Li [103], where the authors solve a continuous-time Riccati equation to compute the value function. In [53] this was extended to include a long-only constraint. Skaf and Boyd [87], and Gârleanu and Pederson [37], point out that the multi-period portfolio optimization problem with linear dynamics and convex quadratic objective can be solved exactly. For problems with more complex objective terms, such as proportional transaction costs, Skaf and Boyd use the value functions for an associated quadratic problem as the approximate value functions in an ADP policy. In [43] the authors formulate a multi-period portfolio problem as a linear stochastic control problem, and propose an MPC policy.

Optimal execution

An important special case of the multi-period problem is the optimal execution problem, where we seek to execute a large block of trades while incurring as small a cost as possible. Bertsimas and Lo [16] model price impact, in which trading affects the asset prices, and derive an optimal trading policy using dynamic programming methods. Almgren and Chriss [4] address the optimal execution problem, including volatility of revenue. They show that the optimal policy can be obtained with additional restrictions on the price dynamics.

Performance bounds

In problems for which an optimal policy can be found, the optimal performance serves as a (tight) bound on performance. The present paper focuses on developing a numerical bound on the optimal performance for problems for which the optimal policy cannot be found.

Brown and Smith [22] compute a bound on optimal performance and derive a heuristic policy that achieves performance close to the bound. Their bound is given by the performance of an investor with perfect information about future returns, plus a clairvoyance penalty.

In [41], the authors construct an upper bound on a continuous time portfolio utility maximization problem with position limits. They do this by solving an unconstrained 'fictitious problem' which provides an upper bound on the value function of the original problem.

In [70], the authors describe a class of linear rebalancing policies for the discrete-time portfolio optimization problem. They develop several bounds, including a bound based on a clairvoyant investor and a bound obtained by solving an unconstrained quadratic problem.

Desai et al. [32] develop a bound for an optimal stopping problem, which is useful in a financial context for the pricing of American or Bermudan derivatives amongst other applications. The bound is derived from a dual characterization of optimal stopping problems as optimization problems over the space of martingales.

1.3 Outline

We structure our paper as follows. In chapter 2 we formulate a general multi-period investment problem as a linear convex stochastic control problem, using somewhat nontraditional state variables, and give examples of (convex) stage cost terms and portfolio constraints that arise in practical investment problems, as well as mentioning some nonconvex terms and constraints that do not fit our model. In chapter 3 we review the dynamic programming solution of the stochastic control problem, including the special case when the stage costs are convex quadratic. In chapter 4 we give our method for finding a performance bound in outline form; the full derivations are pushed to appendices A–C. We describe MPC in chapter 6. In chapter 7 we report numerical results for several examples, using both ADP and MPC trading policies.

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