The Many Faces of Degeneracy in Conic Optimization

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Contents

Abstract

Slater's condition – existence of a "strictly feasible solution" – is a common assumption in conic optimization. Without strict feasibility, first-order optimality conditions may be meaningless, the dual problem may yield little information about the primal, and small changes in the data may render the problem infeasible. Hence, failure of strict feasibility can negatively impact off-the-shelf numerical methods, such as primal-dual interior point methods, in particular. New optimization modeling techniques and convex relaxations for hard nonconvex problems have shown that the loss of strict feasibility is a more pronounced phenomenon than has previously been realized. In this text, we describe various reasons for the loss of strict feasibility, whether due to poor modeling choices or (more interestingly) rich underlying structure, and discuss ways to cope with it and, in many pronounced cases, how to use it as an advantage. In large part, we emphasize the facial reduction preprocessing technique due to its mathematical elegance, geometric transparency, and computational potential.

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What this monograph is about

Conic optimization has proven to be an elegant and powerful modeling tool with surprisingly many applications. The classical *linear programming* problem revolutionized operations research and is still the most widely used optimization model. This is due to the elegant theory and the ability to solve in practice both small and large scale problems efficiently and accurately by the well known simplex method of Dantzig [37] and by more recent interior-point methods for convex and nonconvex problems, e.g., [151, 100, 27]. The size (number of variables) of linear programs that could be solved before the interior-point revolution was on the order of tens of thousands, whereas it immediately increased to millions for many applications. A large part of modern success is due to *preprocessing*, which aims to identify (primal and dual slack) variables that are identically zero on the feasible set. The article [98] is a good reference.

The story does not end with linear programming. Dantzig himself recounts in [38]: "the world is nonlinear". Nonlinear models can significantly improve on linear programs if they can be solved efficiently. Conic optimization has shown its worth in its elegant theory, efficient algorithms, and many applications e.g., [149, 10, 21]. Preprocessing to rectify possible loss of "strict-feasibility" in the primal or the dual problems is appealing for general conic optimization as well. In contrast to linear programming, however, the area of preprocessing for conic optimization is in its infancy; see e.g., [31, 140, 32, 109, 111] and Section 1.1, below. In contrast to linear programming, numerical error makes preprocessing difficult in full generality. This being said, surprisingly, there are many specific applications of conic optimization, where the rich underlying structure makes preprocessing possible, leading to greatly simplified models and strengthened algorithms. Indeed, exploiting structure is essential for making preprocessing viable. In this monograph, we present the background and the elementary theory of such regularization techniques in the framework of *facial reduction (FR)*. We focus on notable case studies, where such techniques have proven to be useful.

1.1 Related work

To put this text in perspective, it is instructive to consider nonlinear programming. Nontrivial statements in constrained nonlinear optimization always rely on some regularity of the constraints. To illustrate, consider a minimization problem over a set of the form $\{x : f(x) = 0\}$ for some smooth f. How general are such constraints? A celebrated result of Whitney [146] shows that *any* closed set in a Euclidean space can written as a zero-set of some C^∞-smooth function f. Thus, in this generality, there is little difference between minimizing over arbitrary closed sets and sets of the form $\{x : f(x) = 0\}$, for smooth f. Since little can be said about optimizing over arbitrary closed sets, one must make an assumption on the equality constraint. The simplest one, eliminating Whitney's construction, is that the gradient of f is nonzero on the feasible region – the earliest form of a constraint qualification. There have been numerous papers, developing weakened versions of regularity (and optimality conditions) in nonlinear programming; some good examples are [64, 26, 23].

The Slater constraint qualification, we discuss in this text, is in a similar spirit, but in the context of (convex) conic optimization. Some good early references on the geometry of the Slater condition, and weakened variants, are [59, 95, 96, 147, 20]. The concept of facial reduction for general convex programs was introduced in [24, 25], while an early application to a semi-definite type best-approximation problem was given in [148]. Recently, there has been a significant renewed interest in facial reduction, in large part due to the success in applications for graph related problems, such as Euclidean distance matrix completion and molecular conformation [78, 77, 48, 6] and in polynomial optimization [112, 113, 76, 144, 143]. In particular, a more modern explanation of the facial reduction procedure can be found in [89, 106, 109, 138, 145].

We note in passing that numerous papers show that strict feasibility holds "generically" with respect to unstructured perturbations. In contrast, optimization problems appearing in applications are often highly structured and such genericity results are of little practical use.

1.2 Outline of the monograph

The monograph is divided into two parts. In Part I, we present the necessary theoretical grounding in conic optimization, including basic optimality and duality theory, connections of Slater's condition to the *distance to infeasibility* and sensitivity theory, the facial reduction procedure, and the singularity degree. In Part II, we concentrate on illustrative examples and applications, including matrix completion problems (semi-definite, low-rank, and Euclidean distance), relaxations of hard combinatorial problems (quadratic assignment and max-cut), and sum of squares relaxations of polynomial optimization problems.

1.3 Reflections on Jonathan Borwein and FR

These are some reflections on Jonathan Borwein and his role in the development of the facial reduction technique, by Henry Wolkowicz. Jonathon Borwein passed away unexpectedly on Aug. 2, 2016. Jon was an extraordinary mathematician who made significant contributions in an amazing number of very diverse areas. Many details and personal memories by myself and many others including family, friends, and colleagues, are presented at the memorial website `jonborwein.org`.

This was a terrible loss to his family and all his friends and colleagues, including myself. The facial reduction process we use in this monograph originates in the work of Jon and the second author (myself). This work took place from July of 1978 to July of 1979 when I went to Halifax to work with Jon at Dalhousie University in a lectureship position. The optimality conditions for the general abstract convex program using the facially reduced problem is presented in the two papers [24, 23]. The facial reduction process is then derived in [25].

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