Multi-Period Trading via Convex Optimization

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Foundations and Trends® in Optimization, 2016, Volume 3, 4 issues. ISSN paper version 2167-3888. ISSN online version 2167-3918. Also available as a combined paper and online subscription.
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Abstract

We consider a basic model of multi-period trading, which can be used to evaluate the performance of a trading strategy. We describe a framework for single-period optimization, where the trades in each period are found by solving a convex optimization problem that trades off expected return, risk, transaction cost and holding cost such as the borrowing cost for shorting assets. We then describe a multi-period version of the trading method, where optimization is used to plan a sequence of trades, with only the first one executed, using estimates of future quantities that are unknown when the trades are chosen. The single-period method traces back to Markowitz; the multi-period methods trace back to model predictive control. Our contribution is to describe the single-period and multi-period methods in one simple framework, giving a clear description of the development and the approximations made. In this paper we do not address a critical component in a trading algorithm, the predictions or forecasts of future quantities. The methods we describe in this paper can be thought of as good ways to exploit predictions, no matter how they are made. We have also developed a companion open-source software library that implements many of the ideas and methods described in the paper.

DOI: 10.1561/2400000023.
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Introduction

Single and multi-period portfolio selection. Markowitz [54] was the first to formulate the choice of an investment portfolio as an optimization problem trading off risk and return. Traditionally, this was done independently of the cost associated with trading, which can be significant when trades are made over multiple periods [49]. Goldsmith [38] was among the first to consider the effect of transaction cost on portfolio selection in a single-period setting. It is possible to include many other costs and constraints in a single-period optimization formulation for portfolio selection [53, 63].

In multi-period portfolio selection, the portfolio selection problem is to choose a sequence of trades to carry out over a set of periods. There has been much research on this topic since the work of Samuelson [74] and Merton [58, 59]. Constantinides [22] extended Samuelson’s discrete-time formulation to problems with proportional transaction costs. Davis and Norman [24] and Dumas and Lucian [30] derived similar results for the continuous-time formulation. Transaction costs, constraints, and time-varying forecasts are more naturally dealt with in a multi-period setting. Following Samuelson and Merton, the literature on multi-period portfolio selection is predominantly based on dynamic
programming \[5\] [9], which properly takes into account the idea of recourse and updated information available as the sequence of trades are chosen (see \[37\] and references therein). Unfortunately, actually carrying out dynamic programming for trade selection is impractical, except for some very special or small cases, due to the ‘curse of dimensionality’ \[72\] [11]. As a consequence, most studies include only a very limited number of assets and simple objectives and constraints. A large literature studies multi-period portfolio selection in the absence of transaction cost (see, e.g., \[18\] and references therein); in this special case, dynamic programming is tractable.

For practical implementation, various approximations of the dynamic programming approach are often used, such as approximate dynamic programming, or even simpler formulations that generalize the single-period formulations to multi-period optimization problems \[11\]. We will focus on these simple multi-period methods in this paper. While these simplified approaches can be criticized for only approximating the full dynamic programming trading policy, the performance loss is likely very small in practical problems, for several reasons. In \[11\], the authors developed a numerical bounding method that quantifies the loss of optimality when using a simplified approach, and found it to be very small in numerical examples. But in fact, the dynamic programming formulation is itself an approximation, based on assumptions (like independent or identically distributed returns) that need not hold well in practice, so the idea of an ‘optimal strategy’ itself should be regarded with some suspicion.

**Why now?** What is different now, compared to 10, 20, or 30 years ago, is vastly more powerful computers, better algorithms, specification languages for optimization, and access to much more data. These developments have changed how we can use optimization in multi-period investing. In particular, we can now quickly run full-blown optimization, run multi-period optimization, and search over hyper-parameters in back-tests. We can run end-to-end analyses, indeed many at a time in parallel. Earlier generations of investment researchers, relying on computers much less powerful than today, relied more on separate models
and analyses to estimate parameter values, and tested signals using simplified (usually unconstrained) optimization.

**Goal.** In this tutorial paper we consider multi-period investment and trading. Our goal is to describe a simple model that takes into account the main practical issues that arise, and several simple and practical frameworks based on solving convex optimization problems [13] that determine the trades to make. We describe the approximations made, and briefly discuss how the methods can be used in practice. Our methods do not give a complete trading system, since we leave a critical part unspecified: Forecasting future returns, volumes, volatilities, and other important quantities (see, e.g., [42]). This paper describes good practical methods that can be used to trade, given forecasts.

The optimization-based trading methods we describe are practical and reliable when the problems to be solved are convex. Real-world single-period convex problems with thousands of assets can be solved using generic algorithms in well under a second, which is critical for evaluating a proposed algorithm with historical or simulated data, for many values of the parameters in the method.

**Outline.** We start in chapter 2 by describing a simple model of multi-period trading, taking into account returns, trading costs, holding costs, and (some) corporate actions. This model allows us to carry out simulation, used for what-if analyses, to see what would have happened under different conditions, or with a different trading strategy. The data in simulation can be realized past data (in a back-test) or simulated data that did not occur, but could have occurred (in a what-if simulation), or data chosen to be particularly challenging (in a stress-test). In chapter 3 we review several common metrics used to evaluate (realized or simulated) trading performance, such as active return and risk with respect to a benchmark.

We then turn to optimization-based trading strategies. In chapter 4 we describe single-period optimization (SPO), a simple but effective framework for trading based on optimizing the portfolio performance over a single period. In chapter 5 we consider multi-period optimiza-
tion (MPO), where the trades are chosen by solving an optimization problem that covers multiple periods in the future.

**Contribution.** Most of the material that appears in this paper has appeared before, in other papers, books, or EE364A, the Stanford course on convex optimization. Our contribution is to collect in one place the basic definitions, a careful description of the model, and discussion of how convex optimization can be used in multi-period trading, all in a common notation and framework. Our goal is not to survey all the work done in this and related areas, but rather to give a unified, self-contained treatment. Our focus is not on theoretical issues, but on practical ones that arise in multi-period trading. To further this goal, we have developed an accompanying open-source software library implemented in Python, and available at

[https://github.com/cvxgrp/cvxportfolio](https://github.com/cvxgrp/cvxportfolio)

**Target audience.** We assume that the reader has a background in the basic ideas of quantitative portfolio selection, trading, and finance, as described for example in the books by Grinold & Kahn [42], Meucci [60], or Narang [65]. We also assume that the reader has seen some basic mathematical optimization, specifically convex optimization [13]. The reader certainly does not need to know more than the very basic ideas of convex optimization, for example the overview material covered in chapter 1 of [13]. In a nutshell, our target reader is a quantitative trader, or someone who works with or for, or employs, one.
References


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References


