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Finite-Time Stability Tools for Control and Estimation

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Contents

Finite-Time Stability Tools for Control and Estimation

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ABSTRACT

This monograph presents some existing and new results on analysis and design of finite-time and fixed-time converging systems. Two main groups of approaches for analysis/synthesis of this kind of convergence are considered: based on Lyapunov functions and the theory of homogeneous systems. The focus is put on the dynamics described by ordinary differential equations, time-delay models and partial differential equations. Some popular control and estimation algorithms, which possess accelerated converge rates, are reviewed. The issues of discretization of finite-/fixed-time converging systems are discussed.

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Introduction and motivation

1.1 General terminology

To design an estimation or a control algorithm we have to select performance criteria to be optimized. Stability is one of the main quality indexes, which is usually investigated with respect to an invariant mode (*e.g.*, an equilibrium, desired trajectory or a set), so another important characteristic is the time of convergence of the system trajectories to this mode, which can be infinite (in linear time-invariant systems) or finite. In the latter case the limit mode has to be exactly established in a *finite time* (dependent on initial deviations). If such a time is independent on initial conditions, then this kind of convergence is called *fixed-time*. The notion of finite-time stability was proposed in the 1960s by Emilo Roxin (Roxin, 1966), and has been developed in many later works, where particular attention has been paid to the time of convergence to a steady state. The fixed-time stability concept is quite recent (Polyakov, 2012). Many results on analysis and design of control and estimation algorithms in this context have been obtained, which profit from a fine development of the Lyapunov function method to these sorts of stability. Another useful and simple method to deal with finite- or fixed-time stability is based on the theory of dynamical homogeneous systems.

Homogeneity is a property of mathematical objects, such as functions or vector fields, to be scaled in a consistent manner with respect to a scaling operation (called a dilation) applied to their argument. Such a type of symmetry is defined in a way leading to the scaling of trajectories of corresponding dynamical systems. Homogeneous systems possess several important and useful properties: their local behavior is the same as global one, the rate of convergence to the origin can be identified by degree of homogeneity (a parameter of symmetry), and the stability is robust to various perturbations. The first rise of homogeneity consists of homogeneous polynomials, investigated by Euler in the 18th century. In the 1950s and 1960s, more generic notions of homogeneity (weighted and coordinate-free or geometric) were introduced by Vladimir Zubov and his group (see, Zubov, 1964; Khomenuk, 1961 and the references therein). Many other studies have been performed in the last decades, and these notions of homogeneity have been extended to other kinds of systems (discontinuous models, time-delay systems, partial differential equations, discrete-time models).

1.2 Historical remarks

The concept of asymptotic stability, introduced by the famous thesis of Lyapunov (1992), is one of the central notions of modern control theory. Many problems of state estimation and control can be reduced to a stability analysis or to a stabilization of solutions of certain dynamical models. As a profound and fundamental concept, the classical notion of asymptotic stability does not characterize a convergence time (known as a time response in control theory) of a nonlinear system, and in such a case it is implicitly assumed that the desired mode is reached when $t \to +\infty$. In Erugin (1951), it was mentioned that certain asymptotically stable systems may reach a stable equilibrium in a finite time. This time can be considered as a quantitative characteristic of the time response of a control system. The notion of finite stability (also known today as finite-time stability, see Bhat and Bernstein, 2000) was introduced in Roxin (1966).

In control theory, finite-time convergence to a set-point was studied in the context of the so-called minimum time control problem (Feldbaum, 1953). The corresponding optimal solution can be obtained in the feedforward form according the celebrated Feldbaum theorem about n intervals. In Fuller (1960), it was discovered that some optimal control problems have solutions in the form of finite-time stabilizing relay feedbacks. The sliding mode control theory still uses the finite-time stabilization properties of relay feedbacks for enforcing a sliding motion (Utkin, 1992; Shtessel *et al.*, 2014). In a more systematic way, the problem of finite-time feedback stabilization has been studied in Haimo (1986) for planar systems. In Korobov (1979) and Korobov (1980), the problem of finite-time stabilization with a bounded control magnitude was studied for linear and nonlinear plants using the so-called controllability function method. Relations of finite-time stability and controllability of nonlinear systems were also studied in Kawski (1989) and Coron and Praly (1991).

Being the main tool for the stability analysis of nonlinear systems, the Lyapunov function method has been utilized for characterization of finite-time stability since Roxin (1966). However, the "modern era" of finite-time stability and stabilization in control was initiated in Bhat and Bernstein (2000), where the Lyapunov function method for finite-time stability analysis was refined. In the last 20 years, finite-time control and estimation algorithms were developed for linear/nonlinear and finite/infinite dimensional models. The aim of this monograph is to survey some existing tools for finite-time stability analysis and finite-time controllers/observers design.

1.3 Motivating examples

Initially, we would like to consider a few motivating examples, where finite-time stability and stabilization can be discovered in system models and/or utilized for control design purposes.

1.3.1 Torricelli's law

The law of fluid dynamics, which was discovered by Italian scientist Evangelista Torricelli in 1643, relates the velocity of fluid flowing from an orifice to the height of the fluid above the opening. The law establishes that the speed of a fluid v exiting through a hole, located at the bottom of the tank and filled by the liquid till the level h, equals

$$v = \sqrt{2gh},$$

where g is the acceleration due to the gravity (there are also mild assumptions on the properties of the orifice and the liquid). This expression can be obtained by equating the gained kinetic energy, $\frac{mv^2}{2}$, with the lost potential energy, mgh, and solving it for v.

Torricelli's law can be further used to determine the dynamics of decay of the liquid's level $h(t)$ in the tank, which with the escaping water naturally goes in time $t \geq 0$ from the initial value $h(0) > 0$ till zero. To this end, assuming that the tank is cylindrical with a fixed cross-sectional area $A = \pi r^2$, where $r > 0$ is the radius of the tank, and the cross-section of the hole located at the bottom of the tank is $a > 0$, then the rate of outflow can be obtained as

$$A\frac{dh(t)}{dt} = -av(t) = -a\sqrt{2gh(t)}$$

while $h(t) > 0$. Hence,

$$\frac{dh(t)}{dt} = -\frac{a}{A}\sqrt{2gh(t)},$$

and the direct computations show that $h(T) = 0$ in a finite time $T = \frac{A}{a}\sqrt{\frac{2}{g}h(0)}$. Such a finite-time escape of water from a volume is a basic experience, which was used by humans in engineering for centuries as, for example, in *clepsydra* (that is a clock measuring the time using the flow of water).

1.3.2 Mechanical models with dry and viscous frictions

Let us consider a mechanical system consisting of a rigid body moving laterally on a contact surface and in a viscous environment (fluid). The simplest real-life example of such a mechanical system is a car moving on a flat road.

Let $z(t)$ be the position of the center of mass of the body in an inertial frame at time $t \in \mathbb{R}$. The equation describing a motion of this system has the form

$$\dot{z}(t) = v(t), \quad m\,\dot{v}(t) = F(t), \quad t > 0, \quad z(t) \in \mathbb{R},$$

where $v(t)$ is the velocity, m is the mass of the body, and F is the sum of external forces.

Let us consider only the deceleration phase of the motion assuming that at the initial instant of time this mechanical system has some non-zero velocity $\dot{z}(0) = v(0) \neq 0$. Dissipation of the energy is caused, basically, by two external forces:

- the *drag force* (fluid resistance) is proportional to the velocity squared (see Falkovich, 2011)

$$F_{drag}(t) = -k_{drag}\, v^2(t)\ \text{sign}(v(t)),$$

  where $k_{drag} > 0$ is the coefficient of fluid (air) resistance and the sign function is given by

$$\text{sign}(\rho) = \begin{cases} 1 & \text{if} \quad \rho > 0, \\ 0 & \text{if} \quad \rho = 0, \\ -1 & \text{if} \quad \rho < 0; \end{cases}$$

- the *dry friction force* is nearly independent of the velocity and can be modeled as follows (see ,*e.g.*, Armstrong-Helouvry, 1991)

$$F_{dry}(t) = -k_{dry}\ \text{sign}(v(t)),$$

  where $k_{dry} > 0$ is the coefficient of dry friction.

A more general friction model also may contain some linear terms (proportional to the velocity). We skip them for simplicity of analysis, since they will not change any conclusion about convergence rates of the system.

The sum of external forces $F(t)$ can be represented as follows

$$F(t) = F_{drag}(t) + F_{dry}(t) = -\left(k_{dry} + k_{drag}\, v^2(t)\right) \text{sign}(v(t)).$$

and the differential equation describing an evolution of the velocity of the body has the form:

$$m\dot{v}(t) = -\left(k_{dry} + k_{drag}\, v^2(t)\right) \text{sign}(v(t)).$$

It is not difficult to show that $v = 0$ is the equilibrium of the latter equation, which is globally asymptotically stable, $v(t) \to 0$ as $t \to +\infty$.

The solution of this ODE can be found explicitly:

$$v(t) = \tan\left(\arctan(|v(0)|) - \frac{\sqrt{k_{dry}\,k_{drag}}}{m}t\right)\operatorname{sign}(v(0)).$$

This immediately implies $v(t) = 0$ for $t \geq \frac{m\arctan(|v(0)|)}{\sqrt{k_{dry}\,k_{drag}}}$. The function arctan is globally uniformly bounded. We conclude that *independently of the initial velocity, the motion of the body terminates no later than the following instant of time*

$$T_{\max} = \frac{m\pi}{2\sqrt{k_{dry}\,k_{drag}}}.$$

This property is also known as the *fixed-time stability* (or the *fixed-time convergence*).

1.3.3 Extinction and blow-up in heat equation

The finite-time operations are omnipresent in the complex dynamical processes described by PDE, and for an example let us consider a reaction-diffusion equation:

$$\tfrac{\partial u(t)}{\partial t} = \Delta u(t) + \varphi(u(t)), \quad t > 0, \quad u(0) = u_0 \in L^2(\mathcal{X}, \mathbb{R}),$$

where $\mathcal{X} \subset \mathbb{R}^n$ is an open connected set with a smooth boundary; $u(t) \in L^2(\mathcal{X}, \mathbb{R})$ is the state of the system supported on $\mathcal{X}$; $t \in \mathbb{R}_+$ is the time variable; the function $u_0 \in L^2(\mathcal{X}, \mathbb{R})$ defines the initial state. The Laplace operator $\Delta = \nabla \cdot \nabla$,

$$\Delta : \mathcal{D}(\Delta) \subset L^2(\mathcal{X}, \mathbb{R}) \to L^2(\mathcal{X}, \mathbb{R}), \quad \mathcal{D}(\Delta) = H^1_0(\mathcal{X}, \mathbb{R}) \cap H^2(\mathcal{X}, \mathbb{R})$$

stands for a characterization of a diffusion, while the function

$$\varphi : L^2(\mathcal{X}, \mathbb{R}) \to L^2(\mathcal{X}, \mathbb{R})$$

expresses reaction/absorption. Let us consider the reaction-diffusion dynamics with a particular nonlinearity:

$$\varphi(u) = k\|u\|^p_{L^2}u$$

with $k \in \mathbb{R}$ and $p > -1$. In this case, the function ϕ is continuously differentiable on $L^2(\mathcal{X}, \mathbb{R})\backslash\{\mathbf{0}\}$, so the system has a unique classical solution $u \in C^1([0,T), L^2(\mathcal{X},\mathbb{R})) : u(t) \in \mathcal{D}, \forall t > 0$ for any initial state $\mathbf{0} \neq u_0 \in \mathcal{D}\backslash\{\mathbf{0}\}$ (see, Pazy, 1983, p. 187). For $u_0 = \mathbf{0}$ the system has the zero solution, which may be non-unique in the general case. Depending on the selection of parameters in the heat equation, a finite-time blow-up can be observed for $k > 0$ and $p > 0$: that is existence of a finite $T_f > 0$ such that

$$\lim_{t\to T_f} \|u(t)\|_{L^2} = +\infty,$$

or a finite-time extinction/absorption can be obtained for $k < 0$ and $p \in (-1, 0)$:

$$\lim_{t\to T_f} \|u(t)\|_{L^2} = 0$$

for some finite $T_f > 0$; finally, if $k < 0$ and $p > 1$, a uniform convergence to a ball in L^2 is recovered

$$\lim_{t\to T_f} \|u(t)\|_{L^2} \leq 1$$

for any $u_0 \in \mathcal{D}$ and some $T_f > 0$. The presented conclusions follow from the identity

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} = -\|\nabla u(t)\|^2_{L^2} + k\|u(t)\|^{p+2}_{L^2}, \quad t > 0,$$

which holds for any classical solution of the considered heat equation.

1.3.4 Minimum time control problem and sliding mode algorithms

A finite-time control algorithm appears in control theory, for example, as a solution of the classical minimum time control problem

$$T \to \min_u$$

subject to

$$\begin{cases} \dot{x}_1 = x_2, & u \in L^\infty((0,T),\mathbb{R}) \text{ such that } |u(t)| \leq 1 \\ \dot{x}_2 = u, & x_1(T) = x_2(T) = 0, \end{cases}$$

Indeed (see, *e.g.*, Chernous'ko *et al.*, 2008), the optimal feedback has the form

$$u = -\operatorname{sign}(|x_2|x_2 + 2x_1),$$

which is, in fact, the so-called high order sliding mode controller (Levant, 2005), which stabilize the origin of the system in a finite time.

1.3.5 Estimation for hybrid systems

In many engineering applications the decisions have to be taken and implemented in a finite time, which may be related with system reconfiguration (*e.g.*, destruction, or disappearance of control/measurements) after a time period. A generic example of such a case is the problem of state estimation (similar consideration can be repeated for the control design) in hybrid systems (or system with impacts or impulses), which can be modeled as

$$\begin{aligned} \dot{x}(t) &= F(t, x(t)) \qquad x(t) \in C, \\ x(t^+) &= G(t, x(t)) \qquad x(t) \in D, \\ y(t) &= H(x(t)), \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $t \geq 0$; the map $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ determines the flow of the system while the movement are continuous and belong to the set $C \subseteq \mathbb{R}^n$, while $G : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ defines the jump or impact dynamics when $x(t)$ enters into the set of discontinuity $D \subseteq \mathbb{R}^n$; $y(t) \in \mathbb{R}^p$ is the output available for measurements. Assume that the form of C, D and the properties of F, G are such that there is no Zeno behavior, *i.e.*, the instants of jumps are always separated in time, and that the solution $x(t)$ of the system is unique and well-defined for any initial condition $x(0) \in \mathbb{R}^n$. It is required to estimate the current state value $x(t)$ using the measured information $y(t)$ and the knowledge about the maps F, G and H, and about the sets C and D.

Note that to design a conventional estimator, which is constituted by the copy of the original dynamics with an output injection in order to adjust the estimate $\hat{x}(t) \in \mathbb{R}^n$:

$$\begin{aligned} \dot{\hat{x}}(t) &= F(t, \hat{x}(t)) + L_C(y(t) - H(\hat{x}(t))) \qquad \hat{x}(t) \in C, \\ \hat{x}(t^+) &= G(t, \hat{x}(t)) + L_D(y(t) - H(\hat{x}(t))) \qquad \hat{x}(t) \in D, \end{aligned}$$

where $L_C, L_D \in \mathbb{R}^{n \times p}$ are the observer gains (for simplicity consider a basic linear correction), it is necessary to make the convergence analysis for the resulting hybrid (nonlinear) dynamics of the estimation error $e(t) = x(t) - \hat{x}(t)$. Such an investigation is rather sophisticated, and if the shape of set D depends on unmeasured variables, then it may be hard to ensure the quality of estimation.

Another solution is in the design of a fast observer that always converges before the impact or reconfiguration. If the original system has an asymptotic convergence rate, then the finite-time decay of $e(t)$ solves the problem. If a fixed-time convergence of $e(t)$ to zero is guaranteed, then uniformity in the uncertainty on initial conditions and the impact influence is provided.

1.3.6 Elimination of the unbounded peaking effect

Any controllable linear system can be stabilized at the origin by means of a static linear feedback. The time of convergence of trajectories from the unit ball into a neighborhood of the origin can be prescribed in advance by means of an appropriate tuning of the feedback gain. Such a stabilization is sufficient for many practical problems. The reasonable question in this case: *Is there any advantage of a nonlinear finite-time stabilizing controller compared to the classical linear feedback?*

Let us consider the control system

$$\dot{x} = Ax + Bu(x), \quad t > 0, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}$$

where $x = (x_1, x_2, ..., x_n)^\top$ is the state vector and $u : \mathbb{R}^n \to \mathbb{R}$ is the feedback control. Initial conditions of the latter system are assumed to be bounded as follows

$$\|x(0)\| \leq 1.$$

The control aim is to stabilize the state vector $x(t)$ of the system into a ball of a small radius $\varepsilon > 0$ in a prescribed time $T > 0$:

$$\|x(t)\| \leq \varepsilon, \quad \forall t \geq T.$$

Let us consider the static linear feedback

$$u_\ell(x) := kx, \quad k = (k_1, k_2, ..., k_n).$$

The eigenvalues $\{\lambda_1, ..., \lambda_n\}$ of the closed-loop linear system

$$\dot{x} = (A + bk)x$$

can be placed in any given set of the complex plane $\mathbb{C}$ by choosing the vector k (see, *e.g.*, Wonham, 1985). Therefore, it is possible to obtain a closed-loop system with an arbitrary fast damping speed, i.e.

$$\forall \varepsilon > 0, \quad \exists k \in \mathbb{R}^{1\times n} : \sup_{\|x_0\|=1} \|x(t)\| < \varepsilon, \quad t > T.$$

Indeed, trajectories of this system converge to the origin exponentially fast

$$\|x(t)\| \leq Ce^{-\sigma t}, t > 0$$

where the constant $C \geq 1$ depends on λ_i, $i = 1, 2, .., n$ and $\Re(\lambda_i) < -\sigma$. Hence, smaller $\varepsilon > 0$ larger $\sigma > 0$ has to be assigned to solve the control problem, i.e. $\sigma \to +\infty$ as $\varepsilon \to 0$ provided that T is fixed. Therefore, we conclude that the linear state feedback is, indeed, a possible solution of the considered stabilization problem for any fixed $\varepsilon > 0$.

However, the trajectories of the closed-loop linear system with fast decays have large deviations from the origin during the initial phase of the stabilization (the constant C depends directly on σ and inversely on ε). This phenomenon is called the "peaking" effect and the large deviation is referred to as an "overshoot" (see Polyak and Smirnov, 2016 for more details). In particular, it is shown (Izmailov, 1987) that there exists $\gamma > 0$ independent of λ_i such that

$$\sup_{0\leq t\leq \sigma^{-1}} \sup_{\|x(0)\|=1} \|x(t)\| \geq \gamma\sigma^{n-1}.$$

Hence, we immediately conclude that a larger $\sigma > 0$ leads to a larger "overshoot" in a shorter time.

For $n > 1$ the linear closed-loop system has an infinite "overshoot" as $\varepsilon \to 0$:

$$\sup_{0\leq t\leq T} \sup_{\|x(0)\|=1} \|x(t)\| \to +\infty \quad \text{as} \quad \varepsilon \to 0.$$

This means that for a sufficiently small $\varepsilon > 0$ the linear system may have so huge "overshoot" that practical (*e.g.*, physical) restrictions to the system states would not allow it. The static linear control needs to be somehow modified to overcome this difficulty. The simplest way is to use some input saturation, which, in fact, must be taken into account anyway in practice. However, in this case it is not clear if the saturated feedback would solve the considered stabilization problem with the prescribed time $T > 0$ provided that the saturation would not destroy stability of the system trajectories initiated in the unit ball.

Another possible way to eliminate the infinite "peaking" effect is a transformation of the linear controller to a finite-time controller. Indeed, let us consider the following feedback law

$$u_h(x) = \tilde{k}\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x,$$

where $\mathbf{d}$ is the weighted dilation

$$\mathbf{d}(s) = \begin{pmatrix} e^{ns} & 0 & \cdots & 0 \\ 0 & e^{(n-1)s} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{s} \end{pmatrix}, \quad s \in \mathbb{R}$$

and $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \to (0, +\infty)$ is the so-called canonical homogeneous norm studied in Section 3.

In Section 4 it is shown that the vector $\tilde{k} = (\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n)^\top$ can be easily selected to guarantee

$$\sup_{\|x(0)\|=1} \|x(t)\| = 0, \quad t \geq T$$

for a fixed $T > 0$. In addition, one can be shown that the feedback law u_h is globally bounded:

$$\sup_{x \in \mathbb{R}^n} |u_h(x)| \leq M < +\infty,$$

where M depends on T as follows: smaller T implies larger M. The homogeneous control stabilizes the considered system globally and in a finite time. It solves the stabilization problem considered above independently of $\varepsilon > 0$. Due to the global boundedness of the controller it does not have the unbounded "peaking" effect discovered for the linear system as $\varepsilon \to 0$.

1.3.7 Separation principle

For a linear system, an output dynamic feedback can be constructed by performing independently any design of observer and controller gains, which is called the *separation principle*. Such a concept simplifies significantly the control development and its implementation, however, it is a well-known fact that the separation principle no longer holds for nonlinear dynamical systems. One reason for that is the insensibility of the independently designed closed-loop system to the converging estimation error, which may create an unstable behavior. In some cases the quality can be seriously damaged by an asymptotically decaying estimator, but the performance can be recovered if the error converges exactly to zero after a finite interval of time, that can be interpreted as validity of the separation principle while using faster than asymptotically (exponentially) converging observers.

For an illustration consider a nonlinear system:

$$\begin{aligned} \dot{z} &= -az + bx_2 z(1 + \ln|z|) + u, \\ \dot{x}_1 &= -k_1 x_1 + x_2, \\ \dot{x}_2 &= -k_2 x_1 + \sin(z), \end{aligned}$$

where $z, x_1, x_2 \in \mathbb{R}$ are the states, a, b, k_1, k_2 are positive parameters, $u \in \mathbb{R}$ is the control input, and

$$y_1 = z, \; y_2 = x_1$$

are the outputs available for measurements.

It is easy to check that the state control

$$u = -bx_2 z(1 + \ln|z|)$$

stabilizes the system globally with an exponential rate of convergence.

Similarly, a conventional high-gain observer can be designed to ensure the estimation of the variable x_2:

$$\begin{aligned} \dot{\hat{x}}_1 &= -k_1 \hat{x}_1 + \hat{x}_2 + \ell_1(y_2 - \hat{x}_1), \\ \dot{\hat{x}}_2 &= -k_2 \hat{x}_1 + \sin(y_1) + \ell_2(y_2 - \hat{x}_1), \end{aligned}$$

where $\hat{x}_1, \hat{x}_2 \in \mathbb{R}$ are the estimates, and ℓ_1, ℓ_2 are positive observer gains. Their tuning ensures a global exponential convergence of the estimation errors $x_i - \hat{x}_i$, $i = 1, 2$ with any desired decay: there exist $M > 0$ and $\sigma > 0$ such that $|x_i(t) - \hat{x}_i(t)| \leq Me^{-\sigma t}E$, $i = 1, 2$ for all $t \geq 0$, where $E = \sqrt{|x_1(0) - \hat{x}_1(0)|^2 + |x_2(0) - \hat{x}_2(0)|^2}$ is an evaluation of the initial deviation.

Here, both the control and the observer are selected independently, and substituting the obtained estimates in the control

$$u = -b\hat{x}_2 y_1(1 + \ln|y_1|),$$

we get the closed-loop dynamics for the variable z (the dynamics of the variables x_1, x_2 are unchanged):

$$\dot{z} = -az + b(x_2 - \hat{x}_2)z(1 + \ln|z|).$$

To evaluate the behavior of z, consider a Lyapunov function $V(z) = |z|$, whose derivative admits an estimate:

$$\dot{V} \leq -aV + \beta E e^{-\sigma t} V(1 + \ln V)$$

for $\beta = bM$, and introducing an auxiliary variable $v = \ln V$ we get:

$$\dot{v} \leq -a + \beta E e^{-\sigma t}(1 + v),$$

whose solutions can be easily calculated analytically:

$$v(t) \leq (v(0) + 1)e^{\frac{\beta E}{\sigma}(1-e^{-\sigma t})} - 1 - a\int_0^t e^{\frac{\beta E}{\sigma}(e^{-\sigma s}-e^{-\sigma t})} ds,$$

or, equivalently,

$$|z(t)| \leq e^{(\ln|z(0)|+1)e^{\frac{\beta E}{\sigma}(1-e^{-\sigma t})} - 1 - a\int_0^t e^{\frac{\beta E}{\sigma}(e^{-\sigma s}-e^{-\sigma t})} ds}.$$

Note that the integral $\int_0^t e^{\frac{\beta E}{\sigma}(e^{-\sigma s}-e^{-\sigma t})} ds$ is a strictly growing unbounded positive function of time $t \geq 0$, and $(\ln|z(0)|+1)e^{\frac{\beta E}{\sigma}(1-e^{-\sigma t})} - 1$ is also an exponentially growing positive function, but asymptotically attaining its maximum $(\ln|z(0)| + 1)e^{\frac{\beta E}{\sigma}} - 1$. The latter value characterizes the peaking overshoot admissible for the variable z and, for example, for $|z(0)| \leq 10$, $E \leq 10$, $b = M = \beta = 1$ and $\sigma = 5$, which are reasonable values of parameters, we get this value of order 10^{10}.

Now assume that a fixed-time converging observer is designed (see Section 5 for the details) such that the estimation error admits a positive definite and radially unbounded Lyapunov function $W(x_1 - \hat{x}_1, x_2 - \hat{x}_2)$ yielding the following estimate on time derivative:

$$\dot{W} \leq -\gamma\left(W^{1-\rho} + W^{1+\rho}\right),$$

where $\gamma > 0$ and $\rho \in (0, 1]$ are design parameters, then

$$W(t) \leq \tan^{\rho^{-1}}\left(\arctan(W^{\rho}(0)) - \rho\gamma t\right)$$

for $t \in [0, \frac{\arctan(W^\rho(0))}{\rho\gamma}] \subseteq [0, \frac{\pi}{2\rho\gamma}]$, and $W(t) = 0$ afterwards. Hence, for the closed-loop system with such an observer, assuming that $\rho = 1$ (without losing generality) the estimate for V can be presented as

$$\dot{V} \leq -aV + b\tan\left(\arctan(E) - \gamma t\right) V(1 + \ln V)$$

for $t \in [0, \frac{\arctan(E)}{\gamma}]$, and

$$\dot{V} \leq -aV$$

for $t \geq \frac{\arctan(E)}{\gamma}$. Obviously, we are interested only in the behavior on the initial interval of time, where using similar steps we get (note that $\int_\tau^t \tan\left(\arctan(E) - \gamma s\right) ds = \gamma^{-1} \ln \frac{|\cos(\arctan(E)-\gamma t)|}{|\cos(\arctan(E)-\gamma\tau)|}$ for any $\tau \leq t$):

$$\begin{aligned}\ln|z(t)| \leq & \ln|z(0)| e^{\frac{b}{\gamma}\ln\frac{|\cos(\arctan(E)-\gamma t)|}{|\cos(\arctan(E))|}} \\ & + \int_0^t e^{\frac{b}{\gamma}\ln\frac{|\cos(\arctan(E)-\gamma t)|}{|\cos(\arctan(E)-\gamma\tau)|}} (b\tan\left(\arctan(E) - \gamma t\right) - a) d\tau\end{aligned}$$

for $t \in [0, \frac{\arctan(E)}{\gamma}]$. On this time interval $\cos\left(\arctan(E) - \gamma t\right)$ is a growing nonnegative function of time, hence, the same is $e^{\frac{b}{\gamma}\ln\frac{|\cos(\arctan(E)-\gamma t)|}{|\cos(\arctan(E))|}}$, but $b\tan\left(\arctan(E) - \gamma t\right) - a$ changes the sign from positive to the negative one, and all these functions are bounded for bounded E (unbounded value is possible for $E = +\infty$ and $t = 0$ only). In this case, again the peaking is governed by the first term $\ln|z(0)|e^{\frac{b}{\gamma}\ln\frac{|\cos(\arctan(E)-\gamma t)|}{|\cos(\arctan(E))|}} = \left(\frac{|\cos(\arctan(E)-\gamma t)|}{|\cos(\arctan(E))|}\right)^{\frac{b}{\gamma}} \ln|z(0)| \leq |\cos\left(\arctan(E)\right)|^{-\frac{b}{\gamma}} \ln|z(0)|$ (proportional to the initial conditions), which is much smaller than the value obtained for the linear high-gain observer. For example, for the same values of parameters in fixed-time scenario the maximum peaking is less than 52, then by tuning the observer parameters it is possible to anticipate such a peaking, while in the case with a parasitic discrepancy of magnitude 10^{10} the gain adjustment cannot help.

1.4 The structure of the monograph

After providing a short historical overview and several motivation examples, this monograph has three main parts dealing with analysis, design and extensions, respectively.

In the first part, the definitions of the different finite-/fixed-time stability properties are given together with their sufficient and necessary characterizations via the Lyapunov function approach. The problem of stabilization with accelerated rates is presented, and robust stability concepts (in the input-to-state stability sense) are described. Next, a detailed presentation of the theory of homogeneous systems is added, and the links with finite-/fixed-time convergence are shown.

In the second part, several stabilization algorithms for linear and nonlinear systems are formalized, which are based on the implicit Lyapunov function approach (an advantage of this method is that the tuning of control parameters can be performed by looking for solutions of linear matrix inequalities). These results are complemented by an observer with accelerated converge rate.

In the third part, the issues of discretization of finite-/fixed-time stable systems are discussed, with a special attention to the solutions obtained with the implicit Lyapunov function method. Finally, the accelerated converge concepts are presented for systems described by time-delay and partial differential equations.

The notation is summarized in the Appendix.

Remark 1.1. The sliding mode control and estimation algorithms are well-known for finite-time convergence behavior (Filippov, 1988; Shtessel *et al.*, 2014), but in this monograph we will mainly focus our attention on continuous systems.

Part III

Discretizations and extensions to infinite dimensional systems

All previously presented results concerning analysis of finite-time and fixed-time stability and convergence properties, or about design of control and estimation algorithms possessing accelerated convergence rates, have been given for continuous-time models. However, if these algorithms have to be realized in digital controllers, for simulations, or in the presence of communication networks, then the considered models should include discrete-time components, time delays or event-based procedures. As we are going to show in this part, the appearance of delays (their can be used to represent all mentioned previously phenomena) drastically changes the abilities of the systems with non-asymptotic convergence. This observation admits a simple intuition: finite-/fixed-time convergences are related with non-Lipschitz behavior and high-gains, and these kinds of dynamics are highly sensitive to any kinds of lags.

This part has two sections. In Section 6, various aspects of discretization of considered finite-time and fixed-time converging homogeneous systems are investigated, and it is demonstrated that the methods of approximation of solutions of this kind of dynamics have to be carefully selected and developed in order to recover the established convergence rates in the discrete time. In Section 7, this discussion is extended by analysis of time-delay systems, and it is concluded by the study of partial differential equations.

6

Implementation and discretization

Usually, for a continuous-time system, after analysis or design have been performed for verification or implementation, the system solutions have to be calculated in a computer or in a digital controller (*e.g.*, for a state observer). For these purposes, different numerical approximation methods and discretization schemes are used (Allen and Isaacson, 1998; Butcher, 2008). For example, the Euler method is a first-order numerical routine for solving ordinary differential equations with a given initial value and time step, which represents the most basic explicit/implicit method of numerical integration and it is the simplest Runge-Kutta method.

6.1 Discretization of homogeneous dynamics

The applicability conditions of most discretization approaches are obtained for locally Lipschitz systems having frequently a local nature. By considering finite-time or fixed-time stable dynamics, we are obliged to deal with non-Lipschitz cases and global comportment. Since homogeneous systems represent a useful case study for finite-time or fixed-time convergences, in this section the implementation issues of stronger-than-asymptotically converging systems and derivation of their solutions will be analyzed for this class of models. First, the applicability of the most popular Euler method will be analyzed. Next, improvements will be presented, which can be obtained by developing consistent discretization tools or state-dependent step methods. Finally, the features of discretization of ILF control and estimation algorithms will be highlighted.

6.1.1 Explicit and implicit Euler methods

The conditions of convergence and stability of the explicit and implicit Euler methods have been studied for linear systems (the notion of A-stability (Butcher, 2008; Dahlquist, 1963)), or for some classes of nonlinear ones. For homogeneous systems it has been shown that application of the explicit Euler method for the global approximation of solutions of homogeneous systems with non-zero degree is problematic (Levant, 2013; Efimov *et al.*, 2017), and the implicit Euler scheme has a better perspective (Acary and Brogliato, 2010; Brogliato and Polyakov, 2015; Huber *et al.*, 2016; Miranda-Villatoro *et al.*, 2017). It is worth stressing that the implicit Euler method has higher computational complexity than the explicit one. Let us present the main statements of these results.

Consider the following nonlinear system:

$$\dot{x}(t) = f(x(t)), \ t \geq 0, \tag{6.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $f : \mathbb{R}^n \to \mathbb{R}^n$ ensures forward existence and uniqueness of the system solutions at least locally (if f is discontinuous, then the solutions are understood in the Filippov's sense (Filippov, 1988)), $f(0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ define the corresponding solution by $X(t, x_0)$ for any $t \geq 0$ for which the solution exists.

In order to approximate solution $X(t, x_0)$ of the system (6.1) for some initial state $x_0 \in \mathbb{R}^n$, select a discretization step $h > 0$, define a sequence of time instants $t_i = ih$ for $i = 0, 1, \ldots$, and denote by x_i an approximation of the solution $X(t_i, x_0)$ at the corresponding time instant (*i.e.*, $x_i \simeq X(t_i, x_0)$ and $x_0 = x(t_0) = x(0)$), then the approximation x_{i+1} calculated in accordance with the explicit Euler method is given by (Butcher, 2008):

$$x_{i+1} = x_i + hf(x_i) \tag{6.2}$$

for $i = 0, 1, \ldots$, while the approximation calculated by the implicit Euler method comes from (Butcher, 2008):

$$x_{i+1} = x_i + hf(x_{i+1}) \tag{6.3}$$

for $i = 0, 1, \ldots$ In the sequel, the problem of convergence to zero of the approximations $\{x_i\}_{i=0}^{\infty}$ derived in (6.2) and (6.3) is studied for the system (6.1) admitting the following hypothesis:

Assumption 6.1. Let (6.1) be $\mathbf{d}$–homogeneous with a degree $\nu \in \mathbb{R}$ and asymptotically stable.

To proceed we need to establish some properties of solutions in (6.2) and (6.3).

Existence of approximations and their relations

Existence of some $x_{i+1} \in \mathbb{R}^n$ for any $x_i \in \mathbb{R}^n$ in the explicit case (6.2) is straightforward, but it is not the case of (6.3). From homogeneity property we can obtain the following result:

Proposition 6.1. (Efimov *et al.*, 2017) Let system (6.1) be $\mathbf{d}$–homogeneous with a degree $\nu \neq 0$. Let for any $x_0 \in S$ and all $h > 0$ there exist a sequence $\{x_i\}_{i=0}^{\infty}$ obtained by (6.2) or (6.3) with initial state x_0. Then for any discretization step $h' > 0$ and for any $y_0 \in \mathbb{R}^n$ there exist a sequence $\{y_i\}_{i=0}^{\infty}$ generated by (6.2) or (6.3) with the step h' and the initial state y_0.

Note that the above result does not provide a conclusion about boundedness or convergence of the obtained sequences.

In the general case, it is difficult to provide some simple conditions for the existence and uniqueness of the solution of the implicit Euler method, but homogeneity may simplify the analysis as usual:

Proposition 6.2. (Efimov *et al.*, 2017) If the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable outside the origin, $\mathbf{d}$–homogeneous of degree $\nu \neq 0$ and there exists $h_0 > 0$ such that

$$\det\left(I_n - h_0 \frac{\partial f(x)}{\partial x}\right) \neq 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \tag{6.4}$$

then for $n \geq 2$ the equation (6.3) has a solution with respect to $x_{i+1} \in \mathbb{R}^n$ for any $x_i \in \mathbb{R}^n$ and for any $h > 0$, additionally, for $n \geq 3$ the solution is unique.

Due to homogeneity there are relations between the approximations obtained for different initial conditions and discretization steps:

Proposition 6.3. (Efimov *et al.*, 2017) Let system (6.1) be $\mathbf{d}$–homogeneous with a degree $\nu \in \mathbb{R}$. If $\{x_i\}_{i=0}^{\infty}$ is a sequence generated by (6.2) or (6.3) with the step h and the initial state x_0, then for any $s \in \mathbb{R}$, $y_i = \mathbf{d}(s)x_i$ is a sequence obtained by (6.2) or (6.3), respectively, with the step $e^{-\nu s}h$ and the initial state $y_0 = \mathbf{d}(s)x_0$.

Note that y_i is an approximation of $X(e^{-\nu s}hi, y_0)$ for shifted instants of time. The following corollaries can be established.

Corollary 6.1. (Efimov *et al.*, 2017) Let system (6.1) be $\mathbf{d}$–homogeneous with a degree $\nu = 0$. Let for all $x_0 \in S$ there exist sequences $\{x_i\}_{i=0}^{\infty}$ obtained by (6.2) or (6.3) with the step $h > 0$ and the initial state x_0 possessing one of the following properties:

$$\sup_{i \geq 0} \|x_i\| < +\infty; \tag{6.5}$$

$$\lim_{i \to +\infty} x_i = 0. \tag{6.6}$$

Then for any $y_0 \in \mathbb{R}^n$ there exist sequences $\{y_i\}_{i=0}^{\infty}$ generated by (6.2) or (6.3) with the step h and the initial state y_0 possessing the same property.

Corollary 6.2. (Efimov *et al.*, 2017) Let system (6.1) be $\mathbf{d}$–homogeneous with a degree $\nu \neq 0$. Let there exist $h > 0$ such that for any $x_0 \in S$ the sequences $\{x_i\}_{i=0}^{\infty}$ obtained by (6.2) or (6.3) with the step h and the initial state x possess one of the properties (6.5), (6.6). Then for any $y_0 \in \mathbb{R}^n$ the sequences $\{y_i\}_{i=0}^{\infty}$ obtained by (6.2) or (6.3) with the step $h\|y_0\|_{\mathbf{d}}^{-\nu}$ and the initial state y_0 possess the same property.

The results of corollaries 6.1 and 6.2 show the advantages and limitations of the Euler method application for calculation of solutions of homogeneous systems with different degrees. For the case $\nu = 0$ the properties of approximation x_i depend on size of the step h, while for $\nu \neq 0$ if a scheme provides approximation of solutions for some h, then similar properties can be obtained for any initial condition with a properly scaled step h'.

Corollary 6.3. (Efimov *et al.*, 2017) Let system (6.1) be $\mathbf{d}$–homogeneous with a degree $\nu \neq 0$. Let for any $x_0 \in \mathbb{R}^n$ and some $h > 0$ there exist sequences $\{x_i\}_{i=0}^{\infty}$ obtained by (6.2) or (6.3) with initial state x_0 possessing one of the properties (6.5), (6.6). Then for any discretization step $h' > 0$ and for any $y_0 \in \mathbb{R}^n$ there exist sequences $\{y_i\}_{i=0}^{\infty}$ generated by (6.2) or (6.3) with the step h' and the initial state y_0 possessing the same property.

Thus, for $\nu \neq 0$ if a scheme provides approximation of solutions *globally* for some h, then similar properties can be obtained for *any* step h'. The latter characteristic is unlikely in general, thus using only homogeneity the global result for the case $\nu \neq 0$ cannot be obtained for (6.2) or (6.3).

Convergence of sequences $\{x_i\}_{i=0}^{\infty}$ generated by Euler methods

Since (6.1) is homogeneous and asymptotically stable under Assumption 6.1, there is a twice continuously differentiable and $\mathbf{d}$–homogeneous Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ of positive degree $\mu > -\nu$ such that

$$\begin{aligned} a &= -\sup_{\xi \in S} L_f V(\xi) > 0, \\ 0 < b &= \sup_{\|\xi\|_{\mathbf{d}} \leq 1} \left\| \frac{\partial V(\xi)}{\partial \xi} \right\| < +\infty, \\ c_1 &= \inf_{\xi \in S} V(\xi),\ c_2 = \sup_{\xi \in S} V(\xi), \\ c_1 \|x\|_{\mathbf{d}}^{\mu} &\leq V(x) \leq c_2 \|x\|_{\mathbf{d}}^{\mu} \quad \forall x \in \mathbb{R}^n. \end{aligned} \tag{6.7}$$

Let us take the discretization step $h > 0$ and consider the behavior of V from (6.7) on a sequence generated by (6.2). For this purpose define $x_i = \mathbf{d}(s)y_i$ with $y_i \in S$ and $s = \ln \|x_i\|_{\mathbf{d}}$:

$$V(x_{i+1}) - V(x_i) = V(x_i + hf(x_i)) - V(x_i)$$
$$= e^{\mu s}[V(y_i + e^{\nu s}hf(y_i)) - V(y_i)] = e^{(\nu+\mu)s}h\frac{\partial V(\xi)}{\partial \xi}f(y_i)$$

for $\xi = y_i + e^{\nu s}\varrho f(y_i)$ with $\varrho \in [0, h]$ and the Mean Value Theorem has been used on the last step. Note that

$$\underline{\sigma}(\|\xi\|_{\mathbf{d}}) \leq \|\xi\| \leq \|y_i\| + \|x_i\|_{\mathbf{d}}^{\nu}\varrho\|f(y_i)\| \leq \overline{\sigma}(1) + g\|x_i\|_{\mathbf{d}}^{\nu}h$$

for $g = \sup_{y\in S}\|f(y)\|$ and some $\underline{\sigma}, \overline{\sigma} \in \mathcal{K}_\infty$. Next,

$$V(x_{i+1}) - V(x_i) = e^{(\nu+\mu)s}h\{\frac{\partial V(y_i)}{\partial y_i}f(y_i)$$
$$+\frac{\partial V(\xi)}{\partial \xi}f(y_i) - \frac{\partial V(y_i)}{\partial y_i}f(y_i)\}$$
$$\leq he^{(\nu+\mu)s}\left(-a + g\left\|\frac{\partial V(\xi)}{\partial \xi} - \frac{\partial V(y_i)}{\partial y_i}\right\|\right).$$

Since $\left\|\frac{\partial V(\xi)}{\partial \xi} - \frac{\partial V(y_i)}{\partial y_i}\right\| \leq k\|\xi - y_i\|$ where $k > 0$ is the Lipschitz constant of $\frac{\partial V(\xi)}{\partial \xi}$ on the set $\xi \in \{\xi \in \mathbb{R}^n : \|\xi\|_{\mathbf{d}} \leq \underline{\sigma}^{-1}(\overline{\sigma}(1) + g\|x_i\|_{\mathbf{d}}^{\nu}h)\}$, then

$$V(x_{i+1}) - V(x_i) \leq he^{(\nu+\mu)s}\{-a + gk\|\xi - y_i\|\}$$
$$\leq he^{(\nu+\mu)s}\{-a + gke^{\nu s}\varrho\|f(y_i)\|\} \leq he^{(\nu+\mu)s}\{-a + g^2ke^{\nu s}h\}.$$

Therefore, the condition of convergence for (6.2) is

$$e^{\nu s}h < \frac{a}{g^2k}, \tag{6.8}$$

where in the right-hand side all constants are independent on the discretization approach. If (6.8) is satisfied, then $V(x_{i+1}) < V(x_i)$, or $\|x_{i+1}\|_{\mathbf{d}} < (c_1^{-1}c_2)^{1/\mu}\|x_i\|_{\mathbf{d}}$.

For the implicit scheme (6.3) exactly the same calculations can be repeated showing that $V(x_{i+1}) < V(x_i)$ under (6.8), then the following results are obtained:

Theorem 6.4. (Efimov *et al.*, 2017) Let Assumption 6.1 hold for $\nu = 0$, then there exists the discretization step $h > 0$ such that the sequences

$\{x_i\}_{i=0}^{\infty}$ obtained by (6.2) or (6.3) for any initial state $x_0 \in \mathbb{R}^n$ and the step h possess the following properties:

(a) $\|x_i\|_{\mathbf{d}} < \gamma \|x_0\|_{\mathbf{d}}$ for all $i \geq 0$ for some $\gamma \geq 1$;

(b) $\lim_{i \to +\infty} x_i = 0$.

Note that for $\nu = 0$ the discrete-time systems are homogeneous in the sense of (Tuna and Teel, 2004), (Sanchez *et al.*, 2019).

Theorem 6.5. (Efimov *et al.*, 2017) Let Assumption 6.1 hold for $\nu < 0$, then for any $\rho > 0$ there exists a discretization step $h_\rho > 0$ such that the sequences $\{x_i\}_{i=0}^{\infty}$ obtained by (6.2) or (6.3) for any initial state $x_0 \in \{x \in \mathbb{R}^n : \|x\|_{\mathbf{d}} > \rho\}$ with a step $h \leq h_\rho$ possess the following properties:

(a) $\|x_i\|_{\mathbf{d}} < \gamma \|x_0\|_{\mathbf{d}}$ for all $i \geq 0$ for some $\gamma \geq 1$;

(b) there exists $i_{x_0} > 0$ such that $x_{i_{x_0}} \in \{x \in \mathbb{R}^n : \|x\|_{\mathbf{d}} \leq \rho\}$.

As follows from Theorem 6.5, in the case $\nu < 0$, for any $h > 0$ the Euler schemes provide the global convergence into some vicinity of the origin, and this vicinity is shrinking as $h \to 0$ (the radius of the vicinity is proportional to $h^{-1/\nu}$ (Levant, 2005)).

Theorem 6.6. (Efimov *et al.*, 2017) Let Assumption 6.1 hold for $\nu > 0$, then for any $\rho > 0$ there exists a discretization step $h_\rho > 0$ such that the sequences $\{x_i\}_{i=0}^{\infty}$ obtained by (6.2) or (6.3) for any initial state $x_0 \in \{x \in \mathbb{R}^n : \|x\|_{\mathbf{d}} \leq \rho\}$ with a step $h \leq h_\rho$ possess the following properties:

(a) $\|x_i\|_{\mathbf{d}} < \gamma \|x_0\|_{\mathbf{d}}$ for all $i \geq 0$ for some $\gamma \geq 1$;

(b) $\lim_{i \to +\infty} x_i = 0$.

According to Theorem 6.6, in the case $\nu > 0$, for any $h > 0$ the Euler schemes preserve the asymptotic convergence to zero locally, and the domain of convergence goes global as $h \to 0$ (it can be shown that the radius ρ is proportional to $h^{-1/\nu}$).

More advantageous conditions for (6.3) can be obtained by imposing some additional but mild restrictions (we also assume that solutions exists, *i.e.*, the conditions of Proposition 6.5 are satisfied):

Theorem 6.7. (Efimov *et al.*, 2017) Let Assumption 6.1 hold and $V : \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable $\mathbf{d}$–homogeneous Lyapunov function of degree μ for the system (6.1). Then for the sequence $\{x_i\}_{i=0}^{\infty}$ generated by the implicit scheme (6.3) with any step $h > 0$ and any $x_0 \in \mathbb{R}^n$, the sequence $\{V(x_i)\}_{i=1}^{+\infty}$ is monotonously decreasing to zero provided that

$$\langle x - y, f(x) \rangle \neq \|x - y\| \cdot \|f(x)\| \tag{6.9}$$

for all $x \neq y$ such that $x, y \in \{z \in \mathbb{R}^n : V(z) = 1\}$.

It is easy to show that if the level set of the Lyapunov function V is convex, then the condition (6.9) of this theorem holds. Note that a time-varying step can be used in the conditions of Theorem 6.7.

It was also proven in Efimov *et al.* (2017) that the explicit Euler scheme (6.2) is divergent for small or big enough initial conditions for the cases $\nu < 0$ or $\nu > 0$, respectively.

Absolute and Relative Errors of Discretized Homogeneous Systems

Standard characteristics of any discretization routine include its precision: an error between the obtained solution approximation x_i and the solution itself $X(ih, x_0)$. In order to evaluate precision of Euler schemes for homogeneous systems, denote by $x_{i+1}(h, x_i)$ the value derived by (6.2) or (6.3) for x_i and $h > 0$, then (Dahlquist and Björck, 2008)

- **absolute error** is the magnitude of the difference between the exact value and its approximation:

$$\Delta(h, x_i) = \|X(h, x_i) - x_{i+1}(h, x_i)\|_{\mathbf{d}};$$

- **relative error** expresses how large the absolute error is compared with the exact value:

$$\delta(h, x_i) = \frac{\Delta(h, x_i)}{\|X(h, x_i)\|_{\mathbf{d}}}.$$

The errors are defined in the homogeneous norm $\|\cdot\|_{\mathbf{d}}$, equivalence of such a formulation and the one based on the conventional norm $\|\cdot\|$ was demonstrated in Efimov *et al.* (2017). The proposed quantities with $\|\cdot\|_{\mathbf{d}}$ suit better for analysis of homogeneous systems:

Theorem 6.8. (Efimov *et al.*, 2017) Let the system (6.1) be $\mathbf{d}$–homogeneous of degree ν and $x_{i+1}(h, x_i)$ be calculated by the explicit (6.2) or implicit (6.3) Euler scheme for $x_i \in \mathbb{R}^n$ and $h > 0$. Then

1) $\Delta(h, \mathbf{d}(s)x_i) = e^s \Delta(he^{\nu s}, x_i)$ and $\delta(h, \mathbf{d}(s)x_i) = \delta(he^{\nu s}, x_i)$ for any $h > 0$ and $x_i \neq 0$;

2) $\delta(h, x_i) \to 0$ as $x_i \to \infty$ if $\nu < 0$;

3) $\delta(h, x_i) \to 0$ as $x_i \to 0$ if $\nu > 0$.

If, in addition, the system (6.1) is asymptotically stable then

4) $\delta(h, x_i) \to \infty$ as $x_i \to 0$ if $\nu < 0$;

5) $\delta(h, x_i) \to \infty$ as $x_i \to \infty$ if $\nu > 0$.

Therefore, for any value of the discretization step, the explicit and implicit Euler schemes provide a good approximation (*i.e.*, small relative error δ) of the system solutions if $\nu < 0$ for big values of initial conditions, and if $\nu > 0$ in a vicinity of the origin. Roughly speaking, if a homogeneous system has a slower rate of convergence than a linear one (far outside of the origin for $\nu < 0$ or in a neighborhood of the origin for $\nu > 0$), then the Euler methods ensure a good precision.

Example 6.1. Consider a scalar stable linear system

$$\dot{x} = -x,$$

which is homogeneous of degree $\nu = 0$ for $\mathbf{d}(s) = e^s$. Its discretizations (6.2) and (6.3) can be written as follows:

$$x_{i+1} = (1 - h)x_i,$$
$$x_{i+1} = (1 + h)^{-1}x_i.$$

Thus, the scheme (6.3) is always converging in this example (by Theorem 6.7, since the system has a Lyapunov function $V = x^2$ for which the condition (6.9) is satisfied), but (6.2) is diverging for any $h > 2$ (an illustration for Corollary 6.1).

Example 6.2. Consider a scalar stable nonlinear system

$$\dot{x} = -|x|x,$$

which is homogeneous of degree $\nu = 1$ for $\mathbf{d}(s) = e^s$, with its corresponding discretizations (6.2) and (6.3):

$$x_{i+1} = (1 - h|x_i|)x_i,$$
$$x_{i+1} = \frac{1}{2h}(\sqrt{4h|x_i| + 1} - 1)\text{sign}(x_i).$$

Then the explicit scheme (6.2) is converging for any $h > 0$ with $|x_i| < 2h^{-1}$ (Theorem 6.6), while the implicit one (6.3) is converging globally (by Theorem 6.7 with $V = x^2$).

Example 6.3. Consider another scalar stable nonlinear system

$$\dot{x} = -|x|^{0.5}\text{sign}(x),$$

which is homogeneous of degree $\nu = -0.5$ for $\mathbf{d}(s) = e^s$, with its corresponding discretizations (6.2) and (6.3):

$$x_{i+1} = (1 - h|x_i|^{-0.5})x_i,$$
$$x_{i+1} = \frac{1}{4}(\sqrt{4|x_i| + h^2} - h)^2\text{sign}(x_i).$$

Then the explicit scheme (6.2) is converging for any $h > 0$ with $|x_i| > 0.25h^2$ (Theorem 6.5), while the implicit one (6.3) is converging globally (again, by Theorem 6.7 with $V = x^2$).

Discussion

For the case of $\nu = 0$ the properties of approximations are dependent on the discretization step h, and convergence to zero of the Euler schemes for one value of the step does not imply the same property for another one (Theorem 6.4). However, for the case $\nu \neq 0$, convergence to the origin or boundedness of approximations obtained for some step on a sphere implies the same property for a properly selected discretization step for any initial condition (Corollary 6.2). In the case of $\nu < 0$ it has been proven that the approximations *globally* converge to some vicinity of the origin (Theorem 6.5). For the case $\nu > 0$, it has been proven that for sufficiently small steps the approximations *locally* converge *in* some vicinity of the origin (Theorem 6.6). The attracting neighborhood of the origin for $\nu < 0$ or the domain of attraction for $\nu > 0$ can be contracted to 0 or enlarged to infinity, respectively, as the step tends to 0.

For the implicit Euler scheme, under additional mild conditions, it has been proven that for any initial conditions and discretization steps the solutions always exist and asymptotically converge to zero (Proposition 6.5 and Theorem 6.7). However, the implicit Euler method has a higher computational complexity than the explicit one. For $\nu < 0$ the explicit Euler method can be used outside of a vicinity of the origin and next switching to the implicit Euler methods is reasonable, in order to demonstrate convergence to the origin (initial application of the explicit method is motivated by its lower computational complexity).

To conclude, the conventional Euler methods do not keep the accelerated convergence rates in discrete time, then other discretization approaches have to be developed for homogeneous systems with nonzero degrees.

6.1.2 Consistent discretization

Motivating examples

Example 6.4. Inspired by Polyakov *et al.* (2019) let us consider the following homogeneous systems:

$$\dot{x} = -2\sqrt{|x|}\overline{\mathrm{sign}}(x), \qquad \overset{y=\sqrt{|x|}\overline{\mathrm{sign}}(x)}{\Leftrightarrow} \qquad \dot{y} \in -\overline{\mathrm{sign}}(y),$$

where

$$\overline{\mathrm{sign}}(\rho) = \begin{cases} 1 & \text{if} \quad \rho > 0, \\ [-1, 1] & \text{if} \quad \rho = 0, \\ -1 & \text{if} \quad \rho < 0. \end{cases}$$

These systems are standard homogeneous, finite-time stable and topologically equivalent (homeomorphic on $\mathbb{R}$ and diffeomorphic on $\mathbb{R}\backslash\{\mathbf{0}\}$). Indeed, if $x(\cdot, x_0)$ is the solution of the first system with $x(0) = x_0 \in \mathbb{R}$ then $y(\cdot, y_0) = \sqrt{|x(\cdot, x_0)|}\,\mathrm{sign}(x(\cdot, x_0))$ is the solution of the second system with $y(0) = y_0 = \sqrt{|x_0|}\,\mathrm{sign}(x_0)$, and vice versa.

The implicit Euler discretizations of these systems are given by:

$$x_{i+1} = x_i - 2h\sqrt{|x_{i+1}|}\,\mathrm{sign}(x_{i+1}) \qquad \not\Leftrightarrow \qquad \begin{array}{l} y_{i+1} = y_i + h\tilde{u}_i \\ \tilde{u}_i \in -\overline{\mathrm{sign}}(y_{i+1}) \end{array}$$

where $h > 0$ is the sampling period, $x_i = x(ih, x_0)$, $y_i = y(ih, y_0)$ for $t \in [ih, (i+1)h)$, and $\tilde{u}_i$ is an auxiliary variable, $i = 0, 1, 2\ldots$. We refer the reader to Acary *et al.* (2012) for more details about the implicit discretization of the discontinuous system.

The discretization destroys the topological equivalence between systems, since

$$x_{i+1} = \left(\sqrt{h^2 + |x_i|} - h\right)^2 \mathrm{sign}(x_i), \quad \not\Leftrightarrow \quad y_{i+1} = \begin{cases} y_i - h\,\mathrm{sign}(y_i) & \text{if } |y_i| > h, \\ 0 & \text{if } |y_i| \le h, \end{cases}$$

Indeed, the discrete-time approximation of the first system is just asymptotically stable ($x_0 \neq 0 \Rightarrow x_i \neq 0, \forall i$), but the discretization of the second equation remains finite-time stable ($\exists i^* = i^*(y_0)$: $y_i = 0, \forall i \geq i^*$). A continuous invertible coordinate transformation, which transforms a solution set of the first discrete-time system to a solution set of the second one, does not exist.

Moreover, the discrete-time approximation of the first (continuous) homogeneous system is inconsistent with its continuous-time counterpart in the context of convergence rates. A reasonable way to discretize it consistently is to use the equivalence with the second system. Using solutions of the consistently discretized (second) system, we can recover the finite-time convergent solutions of the first dynamics by means of the posterior coordinate transformation $\hat{x}_i = y_i^2\,\mathrm{sign}(y_i)$. The suggested approach gives the following approximation

$$\hat{x}_{i+1} = \begin{cases} \left(\sqrt{|\hat{x}_i|} - h\right)^2 \mathrm{sign}(\hat{x}_i) & \text{if} \quad |\hat{x}_i| > h^2, \\ 0 & \text{if} \quad |\hat{x}_i| \le h^2, \end{cases}, \qquad \tilde{x}_0 = x_0$$

which is, obviously, finite-time stable. Any stable continuous homogeneous system of a degree $\mu < 0$ admits *a consistent discrete-time approximation* that preserves the finite-time convergence of all trajectories to the origin.

Example 6.5. Again inspired by Polyakov *et al.* (2019) let us consider the scalar system

$$\dot{x} = -|x|x, \tag{6.10}$$

which is globally nearly fixed-time stable: $|x(t, x_0)| < \varepsilon$ for $t > \frac{1}{\varepsilon}$ independently of the initial state x_0.

The explicit Euler method applied to the system (6.10) gives the discrete-time model

$$x_{i+1} = x_i - h|x_i|x_i.$$

It has solutions which blow up if $h > 2/|x_0|$, i.e. the discrete-time approximation is not globally stable.

The implicit Euler discretization yields the globally asymptotically stable system

$$x_{i+1} = \tfrac{\sqrt{1+4h|x_i|}-1}{2h}\operatorname{sign}(x_i),$$

which does not preserve near fixed-time stability. Thus, it is also inconsistent with the original continuous-time model in the context of the decay rate.

Using a *semi-implicit Euler discretization* we derive

$$\tfrac{x_{i+1}-x_i}{h} = -|x_i|x_{i+1}$$

or, equivalently,

$$x_{i+1} = \tfrac{x_i}{1+h|x_i|}.$$

It is easy see that $|x_1| \le (h)^{-1}$ *independently of* x_0, and

$$|x_2| = \tfrac{1}{|x_1|^{-1}+h} \le \tfrac{1}{h+h} = (2h)^{-1}$$

$$|x_3| = \frac{1}{|x_2|^{-1}+h} \leq \frac{1}{2h+h} = (3h)^{-1},$$

...

$$|x_i| \leq (ih)^{-1}$$

i.e., the obtained discrete-time approximation remains nearly fixed-time stable. In fact, this approach works for any stable **d**-homogeneous system with a positive degree.

Finite-time and Fixed-time stable consistent discretizations

Let us consider a nonlinear system

$$\dot{x} = f(x), \quad t > 0, \quad x(0) = x_0, \tag{6.11}$$

where $x(t) \in \mathbb{R}^n$ is the system state and the nonlinear function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $\mathbb{R}^n \backslash \{\mathbf{0}\}$. Its solutions are understood in the sense of Filippov (see (Filippov, 1988)):

$$\dot{x} \in F(\mathbf{0}) = \bigcap_{\varepsilon>0} \overline{\text{co}} f(B(\varepsilon)\backslash\{\mathbf{0}\}), \tag{6.12}$$

where $\overline{\text{co}}$ denotes a closed convex hull. In our case, $F(x) = \{f(x)\}$ is a singleton for $x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$.

Definition 6.1 (Polyakov *et al.*, 2019)**.** Let $Q : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a (possibly) set-valued mapping. The discrete-time inclusion

$$\mathbf{0} \in Q(h, x_i, x_{i+1}), \quad h > 0, \quad i = 0, 1, 2, ... \tag{6.13}$$

is said to be a *consistent discretization* (discrete-time approximation) of the globally uniformly finite-time stable system (6.11) if

1. for any $\tilde{x} \in \mathbb{R}^n$ and any $h > 0$, there exists $\tilde{x}_h \in \mathbb{R}^n$:

$$\mathbf{0} \in Q(h, \tilde{x}, \tilde{x}_h), \tag{6.14}$$

   and $\tilde{x}_h = \mathbf{0}$ is the unique solution to $\mathbf{0} \in Q(h, \mathbf{0}, \tilde{x}_h)$.

2. for any $h > 0$ each sequence

$$\{x_i\}_{i=0}^{+\infty} \tag{6.15}$$

generated by (6.13) converges to zero in a finite number of steps, i.e., for any $x_0 \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ there exists $i^* > 0$ such that

$$x_i = \mathbf{0} \qquad \text{for} \quad i \geq i^*$$

and $x_{i^*-1} \neq \mathbf{0}$.

3. for any $\varepsilon > 0$ and any $R > \varepsilon$, there exists $\omega \in \mathcal{K}$ such that any sequence (6.15) generated by (6.13) satisfies

$$\|\phi(h, x_i) - x_{i+1}\| \leq h\,\omega(h), \tag{6.16}$$

provided that $\|x_{i+1}\|, \|x_i\| \in [\varepsilon, R]$, where $\phi(\cdot, x_i)$ is a solution to (6.11) with the initial condition $x(0) = x_i$.

The condition 6.16 guarantees that the discrete-time model (6.13) is an approximation of (6.11). Indeed, it defines the one-step discretization error and an approximation error on a time interval $[0, T]$ is $O(\omega(h))$ provided that $h = \frac{T}{N}$, $N \in \mathbb{N}$. This error tends to zero as $h \to 0$ (or, equivalently, $N \to +\infty$). Notice that the approximation errors are defined only on any compact set from $\{x \in \mathbb{R}^n : 0 < \varepsilon \leq \|x\| \leq R < +\infty\}$ due to the singularity of the vector field f at zero.

Definition 6.2 (Polyakov *et al.*, 2019)**.** Let $q : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. The equation

$$q(h, x_i, x_{i+1}) = 0, \quad i = 0, 1, 2, ... \tag{6.17}$$

is said to be a consistent discretization (discrete-time approximation) of the globally nearly fixed-time stable system (6.11) if it satisfies conditions 1 and 3 of Definition 6.1 and for any $r > 0$ there exists $N(r) > 0$ such that any sequence

$$\{x_i\}_{i=0}^{+\infty}, \quad x_0 \neq \mathbf{0} \tag{6.18}$$

generated by the equation (6.17) satisfies

$$\|x_i\| \leq r \qquad \text{for} \quad i \geq N(r)$$

independently of x_0.

Results on consistent discretization

To design a discrete-time approximation for the $\mathbf{d}$-homogeneous ODE (6.11) we use the coordinate transformation (3.22). If f is $\mathbf{d}$-homogeneous of the degree -1, then the right-hand side of the transformed system (3.9) is globally bounded. The following theorem refines the result of Polyakov *et al.* (2019) allowing the condition $f(-x) = -f(x)$ to be omitted. The proof is based on Kakutani fixed-point theorem and can be found in Polyakov (2020).

Theorem 6.9. Let a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous on $\mathbb{R}^n\backslash\{\mathbf{0}\}$, $\mathbf{d}$-homogeneous of the degree -1. Let $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ be the generator of the dilation $\mathbf{d}$ and a symmetric matrix $P \in \mathbb{R}^{n\times n}$ satisfies (3.10).

If the condition (3.12) holds with $\Xi = I_n$, then the mapping $Q : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by

$$Q(h, x_i, x_{i+1}) = \tilde{Q}(h, \Phi(x_i), \Phi(x_{i+1})), \qquad \Phi(x) = \|x\|_{\mathbf{d}}\, \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x, \tag{6.19}$$

where $h > 0$ and

$$\tilde{Q}(h, y_i, y_{i+1}) = y_{i+1} - y_i - h\tilde{F}(y_{i+1}), \qquad \tilde{F}(y) = \bigcap_{\varepsilon>0} \mathrm{co}\, \tilde{f}(y\dot{+}B(\varepsilon)\backslash\{\mathbf{0}\}), \tag{6.20}$$

$$\tilde{f}(y) := \left(\frac{(I-G_{\mathbf{d}})yy^\top P}{y^\top G_{\mathbf{d}} P y} + I_n\right) f\left(\frac{y}{\sqrt{y^\top P y}}\right), \qquad y \in \mathbb{R}^n\backslash\{\mathbf{0}\},$$

defines a consistent discrete-time approximation of the system (6.11) in the sense of Definition 6.1.

The latter theorem is based on the fact that the system $\dot{y} = \tilde{f}(y)$ admits a quadratic Lyapunov function (the condition (3.12) with Ξ =const). However, as was shown in Theorem 3.10, any stable homogeneous system is equivalent to a quadratically stable one. If f in Theorem 6.9 is replaced with the equivalent one:

$$f^{new}(x) = \frac{\partial \Psi(\xi)}{\partial \xi} f(\xi)\Big|_{\xi=\Psi^{-1}(x)}, \quad x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n,$$

where $\Psi \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ is a diffeomorphism on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ given in Theorem 3.10, then the condition $\Xi = I_n$ is fulfilled.

According to Theorem 3.10 a $\mathbf{d}$-homogeneous Lyapunov function $V \in C(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n\backslash\{\mathbf{0}\})$ with the degree 1 can always be found for any asymptotically stable system with a $\mathbf{d}$-homogeneous vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. In this case, the required transformation Ψ can be defined as follows

$$\Psi(\xi) = \mathbf{d}\left(\ln \frac{V(\xi)}{\|\xi\|_{\mathbf{d}}}\right)\xi.$$

Therefore, we obtain the following result.

Corollary 6.10. Any continuous $\mathbf{d}$-homogeneous finite-time stable system

$$\dot{\xi} = f(\xi)$$

with a possible discontinuity at the origin, admits a consistent implicit approximation.

For a positive degree of homogeneity corresponding to the case of nearly fixed-time stability of the origin, we can restrict ourselves to the case of the homogeneity degree 1 without loss of generality. Notice also that the homogeneous vector field with positive degree is always continuous at the origin (see Proposition 3.1).

Theorem 6.11 (Polyakov *et al.*, 2019)**.** *Let a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ be uniformly continuous on S, $\mathbf{d}$-homogeneous of the degree 1 and, as before,*

$$\tilde{f}(z) = \left(\frac{(I-G_{\mathbf{d}})zz^\top P}{z^\top G_{\mathbf{d}} Pz} + I_n\right) f\left(\frac{z}{\|z\|}\right), \quad z \in \mathbb{R}^n\backslash\{\mathbf{0}\},$$

where $G_{\mathbf{d}}$ is the generator of the dilation $\mathbf{d}$, $\|z\| = \sqrt{z^\top Pz}$ and the positive definite matrix $P \in \mathbb{R}^{n\times n}$ satisfies (3.10). If the condition (3.12) holds with $\Xi = I_n$ and

$$S \subset W_\alpha(\mathbb{R}^n) \quad \text{for any} \quad \alpha \in (0, +\infty), \tag{6.21}$$

where $W_\alpha(y) := y - \alpha\|y\|\tilde{f}(y)$, then the function $q : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$q(h, x_i, x_{i+1}) = \tilde{q}(h, \Phi(x_i), \Phi(x_{i+1})), \tag{6.22}$$

where $h > 0$, $\Phi(x) = \|x\|_{\mathbf{d}}\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$ for $x \in \mathbb{R}^n$, and

$$\tilde{q}(h, y_i, y_{i+1}) = y_{i+1} - y_i - h\|y_i\|\,\|y_{i+1}\|\,\tilde{f}(y_{i+1}) \tag{6.23}$$

defines a consistent discrete-time approximation of the practically fixed-time stable system (6.11) *in the sense of Definition 6.2.*

Example 6.6 (Consistent Discretization of Fixed-Time Control in $\mathbb{R}$). Let us consider the scalar sliding mode control system

$$\dot{x} = u, \quad x \in \mathbb{R}$$

with

$$u = \begin{cases} -|x|x & \text{if} \quad |x| > 1, \\ -\operatorname{sign}(x) & \text{if} \quad |x| \leq 1. \end{cases}$$

The control is designed by means of combination of two homogeneous systems with positive and negative degrees. The closed-loop system is globally fixed-time stable, *i.e.*, it is finite-time stable with the globally bounded settling time $T(x_0) \leq T^{\max}$. We have $T^{\max} = 2$ for the considered scalar control system.

Theorems 6.9 and 6.11 can be directly applied to subsystems

$$\dot{x} = -x|x|, \;\; |x| > 1 \qquad \text{and} \qquad \dot{x} = -\operatorname{sign}(x), \;\; |x| \leq 1,$$

which are $\mathbf{d}$-homogeneous of degree -1 and 1, respectively, with $\mathbf{d}(s) = e^{G_{\mathbf{d}}s}$, $s \in \mathbb{R}$ and $G_{\mathbf{d}} = 1$. Applying the consistent discretization schemes to these subsystems we derive

$$x_{i+1} = \begin{cases} \frac{x_i}{1+h|x_i|} & \text{if} \quad |x_i| \geq 1, \\ x_i - h\operatorname{sign}(x_i) & \text{if} \quad h < |x_i| < 1, \\ 0 & \text{if} \quad |x_i| \leq h \end{cases} \quad \begin{matrix} \} - \text{the semi-implicit method} \\ \} - \text{the implicit method} \end{matrix}$$

where $h > 0$ is the sampling period, $i = 0, 1, \ldots$. The obtained discretized model is fixed-time stable, *i.e.*, its trajectories converge to zero in $N(x_0)$ samplings steps, and $N(x_0)$ is globally bounded. Since $N_1(x_0)$ steps ($N_1 \leq 1/h$) are needed to guarantee $|x(th)| = |x_i| \leq 1$ for all $i \geq N_1$ provided that $|x_0| > 1$, and $N_2(x_0)$ steps ($N_2 \leq 1/h$) are needed to reach the origin if $|x_0| \leq 1$, then $N \leq N_1 + N_2$ and

$$N(x_0) \leq \frac{2}{h}.$$

Notice that if the control law is implemented in a sampled way as

$$u(t) = u_i, \quad t \in [ih, (i+1)h),$$

then

$$x(t) = x_i + (t-h)u_i \quad t \in [ih, (i+1)h).$$

In order to guarantee $x((i+1)h) = x_{i+1}$ the control values u_i should be selected as follows

$$u_i = \begin{cases} -\frac{|x_i|x_i}{1+h|x_i|} & \text{if} \quad |x_i| \geq 1, \\ -\operatorname{sign}(x_i) & \text{if} \quad h < |x_i| < 1, \\ -\frac{x_i}{h} & \text{if} \quad |x_i| \leq h, \end{cases} \begin{array}{l} \} \text{ – the semi-implicit method} \\ \} \text{ – the implicit method} \end{array}$$

which correspond to the semi-implicit $u_i = -|x_i|x_{i+1}$ and the implicit $u_i \in -\operatorname{sign}(x_{i+1})$ discretizations of the control law, respectively. Therefore, the closed-loop system with the sampled control is also fixed-time stable, *i.e.*, $x(t) = 0$ for $t \geq hN(x_0)$. This perfectly corresponds to the continuous-time case. Since $hN(x_0) \to T(x_0)$ as $h \to 0^+$, we conclude that the same settling time estimate $hN(x_0) \leq T_{\max} = 2$ holds even for the system with the sampled control actions.

An example of the consistent discretization in $\mathbb{R}^n$ is considered in the Section 6.2.

Discussion

A big advantage of the consistent discretization algorithms is that the obtained discrete-time system with a constant sampling step inherits the convergence rates of the original continuous-time dynamics (*e.g.*, the origin can be reached in a finite number of steps globally in the case with discretization of an FxTS system). This framework can also be used for sampling-time implementation of the controllers providing accelerated convergence to continuous-time plants. The drawback of this method is its computational complexity, since it requires derivation of homogeneous norms on each step, and the consistent discretization always includes implicit methodology, which implies that a nonlinear vector equation has to be solved to calculate x_{i+1}.

6.1.3 Variable-step methods

Following Efimov *et al.* (2019), a state-dependent scaling of the time discretization step is considered in this subsection for explicit and implicit Euler discretization schemes. It is shown that such an approach allows the finite- or fixed-time rates of convergence to be recovered by the discrete-time approximations of solutions, but for an infinite number of steps (in a finite number of steps the convergence to a vicinity of the origin is obtained). It is also demonstrated that relative discretization errors are globally bounded and by decreasing the initial discretization step it is possible to make them arbitrary small, hence, the proposed modification of the Euler method can be indeed used for calculation of solutions of homogeneous dynamics. Comparing to other methods discussed above, utilization of state-dependent discretization steps provides a good compromise between the low computational complexity of this approach and good approximation characteristics.

Euler schemes

As before, consider again a nonlinear system (6.1), where $f : \mathbb{R}^n \to \mathbb{R}^n$ ensures forward existence and uniqueness of the system solutions at least locally, $f(0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ define the corresponding solution by $X(t, x_0)$ for any $t \geq 0$ for which the solution exists. If f is discontinuous, then the solutions are understood in the Filippov's sense (Filippov, 1988).

Let

$$\sup_{\xi \in S} \|f(\xi)\| < +\infty$$

and Assumption 6.1 be satisfied.

To introduce modified Euler schemes with state-dependent step, select a basic discretization step $h > 0$, define a sequence of time instants t_i for $i = 0, 1, \ldots$ such that $t_0 = 0$ and $t_{i+1} - t_i > 0$, and denote by x_i an approximation of the solution $X(t_i, x_0)$ at the corresponding time instant (i.e. $x_i \simeq X(t_i, x_0)$ and $x_0 = X(0, x_0)$), then the approximation x_{i+1} calculated in accordance with the explicit Euler method is given by:

$$x_{i+1} = x_i + \frac{h}{\|x_i\|_{\mathbf{d}}^{\nu}} f(x_i), \tag{6.24}$$
$$t_{i+1} = t_i + \frac{h}{\|x_i\|_{\mathbf{d}}^{\nu}}$$

for $i = 0, 1, \ldots$, while the approximation calculated by the implicit Euler method comes from:

$$x_{i+1} = x_i + \frac{h}{\|x_{i+1}\|_{\mathbf{d}}^{\nu}} f(x_{i+1}), \tag{6.25}$$
$$t_{i+1} = t_i + \frac{h}{\|x_{i+1}\|_{\mathbf{d}}^{\nu}}$$

for $i = 0, 1, \ldots$ In the algorithms (6.24) and (6.25) it is assumed that $x_i \neq 0$ or $x_{i+1} \neq 0$ since in the opposite case the discretization stops at the equilibrium due to $f(0) = 0$.

Remark 6.1. In order to clarify the relations of these schemes with the ones given previously, and also to motivate the selected design, let us consider a function

$$\psi_{x_0}(t) = \int_0^t \|X(s, x_0)\|_{\mathbf{d}}^{\nu} ds$$

for any $x_0 \in \mathbb{R}^n$, which is well-defined and invertible when the trajectory stays out of the origin, *i.e.*, for all $t \in [0, T_0(x_0))$ where $T_0 : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ is possibly infinite time of convergence to the origin. Denote $Y(\tau, x_0) = X(\psi_{x_0}^{-1}(\tau), x_0)$ for $\tau \in [0, \psi_{x_0}(T_0(x_0)))$, then

$$\begin{aligned} \frac{d}{d\tau} Y(\tau, x_0) &= \frac{d}{ds} X(s, x_0)\Big|_{s=\psi_{x_0}^{-1}(\tau)} \frac{1}{\|X(\psi_{x_0}^{-1}(\tau), x_0)\|_{\mathbf{d}}^{\nu}} \\ &= \frac{f(X(\psi_{x_0}^{-1}(\tau), x_0))}{\|X(\psi_{x_0}^{-1}(\tau), x_0)\|_{\mathbf{d}}^{\nu}} = \frac{f(Y(\tau, x_0))}{\|Y(\tau, x_0)\|_{\mathbf{d}}^{\nu}}. \end{aligned} \tag{6.26}$$

Obviously, for any $x_0 \in \mathbb{R}^n$ the asymptotic stability properties of the corresponding solutions $X(t, x_0)$ of the system (6.1) for $t \in [0, T_0(x_0))$ and $Y(\tau, x_0)$ of (6.26) for $\tau \in [0, \psi_{x_0}(T_0(x_0)))$ are interrelated, and the system (6.26) is $\mathbf{d}$–homogeneous with degree 0:

$$\frac{f(\mathbf{d}(s)y)}{\|\mathbf{d}(s)y\|_{\mathbf{d}}^{\nu}} = \mathbf{d}(s) \frac{f(y)}{\|y\|_{\mathbf{d}}^{\nu}}$$

for any $y \in \mathbb{R}^n$ and $s \in \mathbb{R}$. As it has been discussed in the first part of this section (Efimov *et al.*, 2017), for the case $\nu = 0$ and a properly established discretization step $h > 0$ the explicit and implicit Euler methods provide an admissible approximation of the solution $y_i \simeq Y(\tau_i, x_0)$ with $\tau_i = ih$ for $i = 0, 1, \ldots$:

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{\|y_i\|_{\mathbf{d}}^{\nu}} f(y_i), \\ y_{i+1} &= y_i + \frac{h}{\|y_{i+1}\|_{\mathbf{d}}^{\nu}} f(y_{i+1}). \end{aligned}$$

Hence,

$$\tau_{i+1} - \tau_i = h = \int_{t_i}^{t_{i+1}} \|X(s, x_0)\|_{\mathbf{d}}^{\nu} ds,$$

where t_i are the corresponding instants of discretization in the original time t, and under an assumption that $\|X(s, x_0)\|_{\mathbf{d}}^{\nu} \simeq const$ for $s \in [t_i, t_{i+1}]$, *i.e.*, the discretization interval is sufficiently small relative to the velocity of the system, we obtain $t_{i+1} - t_i \simeq \frac{h}{\|X(t_i, x_0)\|_{\mathbf{d}}^{\nu}}$ or $t_{i+1} - t_i \simeq \frac{h}{\|X(t_{i+1}, x_0)\|_{\mathbf{d}}^{\nu}}$ for (6.24) and (6.25), respectively. Above such a scheme is used directly for the system (6.1) to approximate the solution $X(t_i, x_0)$, which is a significant difference from the conventional Euler method. This intuition also highlights the motivation to reduce an Euler computation scheme to a system of degree zero.

One of the main features of (6.24) and (6.25), that is a consequence of this homogeneity property, is as follows:

Proposition 6.4. Let the system (6.1) be $\mathbf{d}$–homogeneous with a degree $\nu \in \mathbb{R}$. If $\{x_i\}_{i=0}^{\infty}$ is a sequence generated by (6.24) or (6.25) for the time instants $\{t_i\}_{i=0}^{\infty}$ with the step h and the initial state $x_0 \in \mathbb{R}$, then for any $s \in \mathbb{R}$, $y_i = \mathbf{d}(s)x_i$ is a sequence obtained by (6.24) or (6.25), respectively, for the instants $e^{-\nu s}\{t_i\}_{i=0}^{\infty}$ with the step h and the initial state $y_0 = \mathbf{d}(s)x_0$.

Note that y_i is an approximation of $X(e^{-\nu s} t_i, y_0)$ for scaled instants of time.

Corollary 6.12. Let the system (6.1) be $\mathbf{d}$–homogeneous with a degree $\nu \in \mathbb{R}$, and for all $x_0 \in S$ there exist sequences $\{x_i\}_{i=0}^{\infty}$ obtained by

(6.24) or (6.25) with the step $h > 0$ possessing one of the following properties:

$$\sup_{i\geq 0} \|x_i\| < +\infty; \tag{6.27}$$

$$\lim_{i\to+\infty} x_i = 0. \tag{6.28}$$

Then for any $y_0 \in \mathbb{R}^n$ there exist sequences $\{y_i\}_{i=0}^{\infty}$ generated by (6.24) or (6.25) with the step h and the initial state y_0 possessing the same property.

In the sequel, the problem of global convergence to zero of the approximations $\{x_i\}_{i=0}^{\infty}$ derived in (6.24) and (6.25) is studied for system in (6.1) satisfying Assumption 6.1 with $\nu \neq 0$ (the case $\nu = 0$, or without scaling of the discretization step, has been analyzed in the first part of this section (Efimov *et al.*, 2017) since it is reduced to the conventional Euler schemes), *i.e.*, we will look for conditions providing the properties (6.27) and (6.28) assumed in Corollary 6.12.

Convergence of sequences $\{x_i\}_{i=0}^{\infty}$ generated by Euler methods

Existence of some $x_{i+1} \in \mathbb{R}^n$ for any $x_i \in \mathbb{R}^n$ in the explicit case (6.24) is straightforward, but it is not the case of (6.25). According to the result of Corollary 6.12, it is enough to find the conditions of existence of x_{i+1} for all $x_i \in S$ in (6.25). Note that in a general case, it is difficult to provide simple conditions for existence and uniqueness of x_{i+1} in the equation (6.25) for any $x_0 \in S$, but as before, the homogeneity may further simplify the solution under additional mild restrictions on f.

Proposition 6.5. Let f be $\mathbf{d}$–homogeneous of degree $\nu \in \mathbb{R}$ and continuous on S. Then there is $h_0 > 0$ such that for all $h \in (0, h_0)$ the equation (6.25) for any $x_i \in \mathbb{R}^n$ has a solution $x_{i+1} \in \mathbb{R}^n$.

Thus, for the proposed implicit Euler method with state-dependent discretization step (6.25), the global existence of solutions may be guaranteed by selecting h sufficiently small provided that f is homogeneous and continuous. Then the principal statement of this subsection is as follows:

Theorem 6.13. Let Assumption 6.1 be satisfied, then there exists $h_0 > 0$ such that for any discretization step $h \in (0, h_0]$ the sequences $\{x_i\}_{i=0}^{\infty}$ obtained by (6.24) or (6.25) for any initial state $x_0 \in \mathbb{R}^n$ and the step h possess the following properties:

(a) $\sup_{i=0,1,\ldots} \|x_i\|_r < \gamma \|x_0\|_r$ for some $\gamma \in (0, +\infty)$;

(b) $\lim_{i\to+\infty} \|x_i\|_r = 0$;

(c) for $\nu = 0$ the sequence $\{x_i\}_{i=0}^{\infty}$ has an exponential convergence rate; for $\nu < 0$ for any $x_0 \in \mathbb{R}^n$ the time of convergence to the origin $t_{+\infty}^{x_0} = \lim_{i\to+\infty} t_i$ is finite; and for $\nu > 0$ the time of convergence from any initial conditions $x_0 \in \mathbb{R}^n$ to $B(\rho)$ with any $\rho > 0$ is also finite independently of x_0.

If the matrix $\frac{\partial^2 V(\xi)}{\partial \xi^2} \geq 0$ for all $\xi \in \mathbb{R}^n$, where $V : \mathbb{R}^n \to \mathbb{R}_+$ is a $\mathbf{d}$–homogeneous Lyapunov function for (6.1), then $h_0 > 0$ can be selected arbitrarily for (6.25).

The requirement on nonnegative definiteness of the second derivative of V is related with the condition of convexity level set of V imposed also in the first part of this section (Efimov *et al.*, 2017).

Absolute and Relative Errors

Denote $\Xi_h(x_0) = X(h\|x_0\|_r^{-\nu}, x_0)$ as the value of the solution of (6.1) with the initial condition $x_0 \in R^n$ evaluated in (6.24) or (6.25) after one iteration with the discretization step $h > 0$ (at $t_1 = h\|x_0\|_r^{-\nu}$ with $t_0 = 0$). Denote by $\hat{\Xi}_h(x_0)$ the estimated value derived by (6.24) or (6.25) for the same x_0 and $h > 0$ (note that $\hat{\Xi}_h(x_0) = x_0 + \frac{h}{\|x_0\|_r^{\nu}} f(x_0)$ in the case of (6.24)), then recall the definitions of the approximation errors in this case:

- the absolute error is the magnitude of the difference between the exact value and its approximation:

$$\Delta^h(x) = \|\Xi_h(x) - \hat{\Xi}_h(x)\|_{\mathbf{d}};$$

- the relative error expresses how large the absolute error is compared with the exact value:

$$\delta^h(x) = \frac{\Delta^h(x)}{\|\Xi_h(x)\|_{\mathbf{d}}}.$$

These error functions admit the following useful properties:

Theorem 6.14. Let the system (6.1) be $\mathbf{d}$–homogeneous of degree $\nu \in \mathbb{R}$ and $\hat{\Xi}_h(x)$ be calculated by the explicit (6.24) or implicit (6.25) Euler scheme for $x \in \mathbb{R}^n$ and $h > 0$. Then the functions $\Delta^h(x)$ and $\delta^h(x)$ are $\mathbf{d}$–homogeneous of degree 1 and 0, respectively.

Any homogeneous function of degree 0 is globally bounded (it may be discontinuous) if its maximal amplitude is finite being evaluated on S. Therefore, if for any initial conditions $x \in S$ the error $\delta^h(x)$ stays sufficiently small for a reasonable selection of h (*i.e.*, the one step error of usual Euler discretization approaches is small on the sphere), then the explicit (6.24) and the implicit (6.25) Euler schemes provide a uniformly bounded relative error δ^h globally. Boundedness of δ^h implies that the difference between $\Xi_h(x)$ and $\hat{\Xi}_h(x)$ stays of the order $\Xi_h(x)$ (roughly speaking proportional to x). Indeed, assume that f is continuously differentiable on S. In this scenario, the second derivative of the solution $X(t, x_0)$ exists and continuous, and for all $x_0 \in S$:

$$\begin{aligned}\Xi_h(x_0) = X(h\|x_0\|_{\mathbf{d}}^{-\nu}, x_0) &= x_0 + h\dot{X}(0, x_0) + \frac{h^2}{2}\ddot{X}(\theta, x_0)\\ &= x_0 + hf(x_0) + \frac{h^2}{2}\ddot{X}(\theta, x_0),\end{aligned}$$

where $\theta \in [0, h]$, and the Lagrange reminder of Taylor series expansion was used. Then for $x_0 \in S$ and the explicit discretization algorithm (6.24):

$$\begin{aligned}\Delta^h(x_0) &= \|\Xi_h(x_0) - \hat{\Xi}_h(x_0)\|_{\mathbf{d}}\\ &= \|x_0 + hf(x_0) + \frac{h^2}{2}\ddot{X}(\theta, x_0) - x_0 - hf(x_0)\|_{\mathbf{d}}\\ &= \frac{h^2}{2}\|\ddot{X}(\theta, x_0)\|_{\mathbf{d}}\end{aligned}$$

for $h \leq 1$ and $\lambda_{\max}(G_{\mathbf{d}}) \leq 1$ (always can be assumed without loosing generality). By differentiability of f, there exists a constant $\varsigma > 0$ such that

$$\sup_{\theta \in [0,1],\, x_0 \in S} \|\ddot{X}(\theta, x_0)\|_{\mathbf{d}} \leq \varsigma,$$

hence,

$$\Delta^h(x_0) \leq \frac{\varsigma}{2}h^2$$

for all $x_0 \in S$ and $h \in (0, 1]$. The quadratic in h convergence of the absolute error $\Delta^h(x_0)$ (the same can be shown for the relative error $\delta^h(x_0)$) implies approaching of the real solution $X(h, x_0)$ by its approximation given in (6.24) (Dahlquist and Björck, 2008). This is an interesting and important observation motivating the use of (6.24) and (6.25) in the applications.

Example 6.7. Consider a scalar stable nonlinear system

$$\dot{x} = -|x|x,$$

which is homogeneous of degree $\nu = 1$ for $\mathbf{d}(s) = e^s$, with its corresponding discretizations (6.24) and (6.25):

$$x_{i+1} = (1-h)x_i,\ t_{i+1} = t_i + \frac{h}{|x_i|};$$
$$x_{i+1} = \frac{x_i}{1+h},\ t_{i+1} = t_i + \frac{h}{|x_{i+1}|},$$

which have a similar form and properties for this case. Both schemes are converging, (6.24) for $h \in (0, 1)$ and (6.25) for any $h > 0$ (Theorem 6.13, and the function $V(x) = x^2$ can be used with positive second derivative).

Example 6.8. Consider another scalar stable nonlinear system

$$\dot{x} = -|x|^{0.5}\text{sign}(x),$$

which is homogeneous of degree $\nu = -0.5$ for $\mathbf{d}(s) = e^s$, with its corresponding discretizations (6.24) and (6.25):

$$x_{i+1} = (1-h)x_i,\ t_{i+1} = t_i + \sqrt{|x_i|}h;$$
$$x_{i+1} = \frac{x_i}{1+h},\ t_{i+1} = t_i + \sqrt{|x_{i+1}|}h,$$

where the update law for x_i has the same form as in the previous example, and only the time instants t_i are scheduled differently (it is not a surprising observation, since the discrete-time dynamics (6.24) and (6.25) are homogeneous of zero degree, then for scalar homogeneous systems (6.1) they have to produce similar expressions). The same convergence properties follow.

Remark 6.2. Note that the time step in (6.24) and (6.25) is state- and degree-dependent. In particular, if $\|x_i\|_{\mathbf{d}} \gg 1$ (it is sufficiently big) and $\nu < 0$, then $t_{i+1} - t_i = \|x_i\|_{\mathbf{d}}^{-\nu} h \gg h$ in (6.24) and the time step can be too large, which is also related with the obtained accuracy estimates in Theorem 6.14. It is worth stressing that the convergence of these algorithms is not influenced, and this observation deals only with the sampling of discretization. Therefore, for big amplitudes of x_i in the case of $\nu < 0$ it is admissible to use the conventional Euler methods (without a scaling of the time discretization step), which may provide a reasonable accuracy of approximation of the solutions under a more regular sampling (Efimov *et al.*, 2017; Levant *et al.*, 2016).

Robustness with respect to external inputs

Let us consider a version of the system (6.1) extended by external inputs:

$$\dot{x}(t) = F(x(t), u(t)),\ t \geq 0, \tag{6.29}$$

with $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $F(x, 0) = f(x)$, where $u : \mathbb{R}_+ \to \mathbb{R}^m$ is a measurable and essentially bounded function of time. For $x_0 \in \mathbb{R}^n$ and an input $u : \mathbb{R}_+ \to \mathbb{R}^m$ denote a corresponding solution of the system (6.29) as $X(t, x_0, u)$. The following hypothesis will be imposed on (6.29):

Assumption 6.2. There exist monotone dilations $\mathbf{d}$, $\tilde{\mathbf{d}}$ and $\nu \in \mathbb{R}$ such that

$$F(\mathbf{d}(s)x, \tilde{\mathbf{d}}(s)u) = e^{\nu s}\mathbf{d}(s)F(x, u)$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $s \in \mathbb{R}$, and there exists $\sigma \in \mathcal{K}_\infty$ such that

$$\sup_{y \in S} \|F(y, u) - F(y, 0)\| \leq \sigma(\|u\|_{\tilde{\mathbf{d}}})$$

for all $u \in \mathbb{R}^m$.

If Assumption 6.1 is also satisfied, then since $F(x,0) = f(x)$, the degrees ν in these assumptions coincide. Moreover, if assumptions 6.1 and 6.2 are both verified, then the system (6.29) possesses the input-to-state stability (ISS) property with respect to u (Bernuau *et al.*, 2013), and the zone of asymptotic convergence (the asymptotic gain) can be evaluated as

$$\|x\|_{\mathbf{d},\infty} \leq \mu \|u\|_{\tilde{\mathbf{d}},\infty},$$

where the gain $\mu > 0$ and

$$\|u\|_{\tilde{\mathbf{d}},\infty} = \operatorname*{ess\,sup}_{t \geq 0} \|u(t)\|_{\tilde{\mathbf{d}}}.$$

Let us show that (6.24) and (6.25) preserve the same stability performance for (6.29). Note that being applied to (6.29) these algorithms can be formulated as follows:

$$x_{i+1} = x_i + \frac{h}{\|x_i\|_{\mathbf{d}}^{\nu}} F(x_i, u_i), \qquad t_{i+1} = t_i + \frac{h}{\|x_i\|_{\mathbf{d}}^{\nu}} \tag{6.30}$$

or

$$x_{i+1} = x_i + \frac{h}{\|x_{i+1}\|_{\mathbf{d}}^{\nu}} F(x_{i+1}, u_{i+1}), \qquad t_{i+1} = t_i + \frac{h}{\|x_{i+1}\|_{\mathbf{d}}^{\nu}} \tag{6.31}$$

for $i = 0, 1, \dots$, respectively, where x_i is an estimate of $X(t_i, x_0, u)$ as before and $u_i = u(t_i)$.

Theorem 6.15. Let assumptions 6.1 and 6.2 be satisfied, then there exists $h_0 > 0$ such that for any discretization step $h \in (0, h_0]$ the discrete-time systems (6.30) or (6.31) are ISS (see (Jiang and Wang, 2001) for the definition of this property and also for its equivalent Lyapunov characterizations for discrete-time systems), and for any initial state $x_0 \in \mathbb{R}^n$ and any bounded input $u : \mathbb{R}_+ \to \mathbb{R}^m$ the corresponding sequences $\{x_i\}_{i=0}^{\infty}$ enter in the set where

$$\sup_{i \geq 0} \|x_i\|_{\mathbf{d}} \leq \mu \sup_{i \geq 0} \|u_i\|_{\tilde{\mathbf{d}}}.$$

Therefore, the proposed schemes (6.24) and (6.25) (or (6.30) and (6.31)) for approximation of solutions of a homogeneous system (6.29) keep unchanged after discretization the convergence rates and stability margins in the presence of perturbations.

Remark 6.3. Note that for an input $u(t) \neq 0$ the corresponding trajectory may visit the origin, but does not stay there due to disturbance presence. In such a case the algorithms (6.30) and (6.31) will decrease to zero the amplitude of the time step for the systems with negative degree or augment it till infinity for the systems with positive degree following their construction. In order to avoid this issue, the upper and lower limits $0 < \underline{h} < \overline{h} < +\infty$ on $h\|x_i\|_{\mathbf{d}}^{-\nu}$ or $h\|x_{i+1}\|_{\mathbf{d}}^{-\nu}$ must be imposed in (6.30) and (6.31), respectively.

Discussion

For a suffisiently small value of h, the Euler methods (6.24) and (6.25) provide a good approximation of the system solutions (Theorem 6.14). For the implicit Euler scheme, under a mild constraint, it has been proven that solutions always exist and converge globally to zero for any h if the Hessian of the Lyapunov function of the system is non-negative definite (Theorem 6.13). Both approaches, (6.24) and (6.25) ((6.30) and (6.31)), preserve the input-to-state stability of the homogeneous dynamics (6.29) with the asymptotic gain of the continuous-time counterpart (Theorem 6.15). The drawback of these approaches is that finite/fixed-time convergence is recovered in infinite number of steps.

6.1.4 Conclusion and comparison of given discretization tools

For Euler algorithms (with constant (6.2), (6.3) or state-dependent (6.24), (6.25) steps), homogeneity simplifies analysis of properties of the obtained discrete approximations of solutions. For example, a certain scalability between approximations calculated for different initial conditions and discretization steps is established in propositions 6.3 and 6.4.

Application of the conventional explicit Euler method (6.2) leads to local approximations of solutions of homogeneous systems with non-

Table 6.1: Comparison of different discretization approaches

	(6.2)	(6.3)	(6.24)	(6.25)	Consistent
Domain	Local	Global	Global	Global	Global
Rates	Asymptotic	Asymptotic	Finite/Fixed	Finite/Fixed	Finite/Fixed
Number of steps	Infinite	Infinite	Infinite	Infinite	Finite
Complexity	+	++	+	++	+++

zero degrees (having finite-time or fixed-time rates of convergence). To calculate global approximations, the implicit method (6.3) can be used, but still asymptotic convergence to the origin is guaranteed in infinite number of steps.

Introducing the state-dependent step in (6.24), (6.25), the obtained approximations can be made globally converging for asymptotically stable homogeneous systems provided that h is selected sufficiently small. In addition, the finite/fixed-time rates of convergence can be guaranteed in infinite number of iterations (Theorem 6.13). In comparison with the standard implicit Euler method (6.3) (having a similar performance), the advantages are that the computationally simpler explicit method (6.24) can be used, and that due to time scaling the approximations have indeed accelerated convergence rates.

The consistent discretization approach gets its name since it provides the finite/fixed-time convergence rates in discrete-time for the solution approximations, and it can also be used for control implementation. It is also the most demanding approach in computations.

Summarizing this discussion, the mentioned properties are briefly described in Table 6.1.

6.2 Digital Implementation of ILF-based algorithms

6.2.1 Practical realization in the form of a linear switched feedback

As discussed in Section 4.2, in order to implement an implicit homogeneous control in practice, a numerical algorithm for a computation of the canonical homogeneous norm is required. This norm can be computed explicitly for $n \leq 2$ approximated by an explicit homogeneous norm for $n \geq 3$. However, even for the second order case the analytical representation of the canonical homogeneous norm is rather cumbersome, so a digital realization of the homogeneous control law requires more computational power than the linear algorithm. Therefore, a sufficiently simple computational algorithm is required for its successful digital realization. Some additional properties of the implicit homogeneous controller are established below for this purpose.

Recall that a non-empty compact set $\Omega \subset \mathbb{R}^n$ is said to be a strictly positively invariant for a dynamical system if $x(t_0) \in \Omega \Rightarrow x(t) \in \operatorname{int}\Omega, t \geq t_0$, where x denotes a trajectory of the dynamical system and $\operatorname{int}\Omega$ denotes the interior of Ω.

Theorem 6.16 (Polyakov, 2020)**.** If all conditions of Corollary 4.6 hold then for any fixed $r > 0$ the closed $\mathbf{d}$-homogeneous ball $\overline{B}_{\mathbf{d}}(r)$ is a strictly positively invariant compact set of the closed-loop system (4.25) with the linear control

$$u_r(x) = K_0 x + r^{1+\mu} K \mathbf{d}(-\ln r) x. \tag{6.32}$$

Now we assume that the value $\|x(t)\|_{\mathbf{d}}$ in the implicit homogeneous controller can be changed only in some sampled instances of time. Let us show that the corresponding linear switched feedback robustly stabilize the perturbed linear system.

Corollary 6.17 (Polyakov *et al.*, 2016b, Polyakov, 2020)**.** If

1) the conditions of Corollary 4.6 hold;

2) $\{t_i\}_{i=0}^{+\infty}$ is an arbitrary sequence of time instances such that

$$0 = t_0 < t_1 < t_2 < \ldots \quad \text{and} \quad \lim_{i\to+\infty} t_i = +\infty;$$

3) the linear switched control u has the form

$$u(x(t)) = K_0 x(t) + \|x(t_i)\|_{\mathbf{d}}^{1+\mu} K \mathbf{d}(-\ln \|x(t_i)\|_{\mathbf{d}}) x(t), \quad t \in [t_i, t_{i+1}) \tag{6.33}$$

then the closed-loop system (4.25), (6.33) is globally asymptotically stable.

The linear switched control (6.33) is obtained from the nonlinear homogeneous one. It can be utilized, for example, in the case when the control system is already equipped with a linear (e.g. analog) controller allowing a change of feedback gains with some sampling period.

According to the proven corollary, the proposed sampled-time realization of the implicit homogeneous controller guarantees asymptotic stabilization of the closed-loop system *independently of the dwell time* (a time between two sampling instants). Such property is not usual for sampled and switched control systems with additive disturbances (Liberzon, 2003). However, without an assumption on the dwell-time we cannot estimate the convergence rate of this system. Obviously, if the dwell time tends to zero the convergence rate tends to the rate of the original continuous system.

To implement the obtained switched linear feedback through these algorithms, an on-line computation of the canonical homogeneous is required. Fortunately, rather simple numerical procedures can be utilized for this purpose.

Let $0 = t_0 < t_1 < t_2 < ...$ be an arbitrary sequence of time instants, $\lim t_i = +\infty$, and $a > 0$, $b > 0$ be the tuning parameters.

Algorithm 6.1 (Polyakov *et al.*, 2015a, Polyakov, 2020).

Initialization $\underline{V} = a$; $\overline{V} = b$; $N_{\max} \in \mathbb{N}$;

Step:

if $x^\top(t_i)\mathbf{d}^\top(-\ln \overline{V})P\mathbf{d}(-\ln \overline{V})x(t_i) > 1$ then

$\underline{V} = \overline{V}$; $\overline{V} = \min(b, 2\overline{V})$;

elseif $x^\top(t_i)\mathbf{d}^\top(-\ln \underline{V})P\mathbf{d}(-\ln \underline{V})x(t_i) < 1$ then

$\overline{V} = \underline{V}$; $\underline{V} = \max(0.5\underline{V}, a)$;

else

for $i = 1 : N_{\max}$

$V = \frac{\underline{V}+\overline{V}}{2}$;

if $x^\top(t_i)\mathbf{d}^\top(-\ln \underline{V})P\mathbf{d}(-\ln \underline{V})x(t_i) < 1$ then

$\overline{V} = V$;

else $\underline{V} = V$;

endif;

endfor;

endif;

$\|x(t_i)\|_{\mathbf{d}} \approx \overline{V}$;

Let $x(t_i) \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ be a given vector and $a = 0$,$b = +\infty$. If the **Step** of the presented algorithm is applied recurrently many times to the same $x(t_i)$ then Algorithm 6.1 guarantees:

1) a localization of the unique positive root of the equation

$$\|\mathbf{d}(-\ln V)x(t_i)\| = 0$$

with respect to $V > 0$, i.e. $V \in [\underline{V}, \overline{V}]$;

2) improvement of the obtained localization by means of the bisection method, i.e. $(\overline{V} - \underline{V}) \to 0$ as number of steps tends to infinity.

Such an application of Algorithm 6.1 allows us to calculate $V \approx \|x(t_i)\|$ with rather high precision, but it requires a high computational capability of a digital device. If the computational power is very restricted, then the **Step** of Algorithm 6.1 may be realized just once at each sampled instant of time. The practical stability of the closed-loop system can be guaranteed in this case. Indeed, Theorem 6.16 proves that the $\mathbf{d}$-homogeneous ball $\bar{B}_{\mathbf{d}}(\overline{V})$ is a strictly positively invariant set of the the closed-loop system with the control $u(x) = \overline{V}^{\mu} K\mathbf{d}(-\ln \overline{V}))$. If the root of the equation $\|\mathbf{d}(-\ln V)x(t_i)\| = 0$ is localized (i.e. $\|x(t_i)\|_{\mathbf{d}} \leq \overline{V}$), Algorithm 6.1 always selects an upper estimate of V to guarantee $x(t_i) \in \bar{B}_{\mathbf{d}}(\overline{V})$. This means that $\|x(t_i)\|_{\mathbf{d}}$ never leaves the ball $\bar{B}_{\mathbf{d}}(\overline{V})$ even when $x(t)$ varies in time.

6.2.2 Consistent discretization of ILF-based algorithms

Let us consider the linear system with the implicit homogeneous control (4.11) recalled here as follows

$$\dot{x} = f(x) := Ax + Bu, \tag{6.34}$$

$$u_\nu(x) := K_0 x + \|x\|_{\mathbf{d}}^{1+\nu} K\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x, \tag{6.35}$$

where $x = (x_1, x_2, ..., x_n)^\top$, $A \in \mathbb{R}^{n\times n}$ is a $\mathbf{d}$-homogeneous matrix of a degree $\nu \in \{-1, 1\}$, the matrix $B \in \mathbb{R}^{n\times m}$ is such that the pair $\{A, B\}$ is controllable, $K \in \mathbb{R}^{m\times n}$ is the matrix of control gains, $\mathbf{d}$ is a dilation in $\mathbb{R}^n$ and $K_0 \in \mathbb{R}^{m\times n}$ is such that the matrix $A_0 = A + BK_0$ is nilpotent.

The closed-loop system is homogeneous of the degree ν:

$$f(\mathbf{d}(s)x) = A\mathbf{d}(s)x + Bu_\nu(\mathbf{d}(s)x) = e^{\nu s}\mathbf{d}(s)(Ax + Bu_\nu(x)) = e^{\nu s}\mathbf{d}(s)f(x).$$

The equivalent transformed homogeneous system with

$$y = \|x\|_{\mathbf{d}}\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$$

has the form

$$\dot{y} = \|y\|^{1+\nu} \left(\frac{(I_n - G_{\mathbf{d}}) y y^\top P}{y^\top P G_{\mathbf{d}} y} + I_n \right) f\left(\frac{y}{\|y\|} \right),$$

where

$$f\left(\frac{y}{\|y\|} \right) = (A + BK) \frac{y}{\|y\|},$$

and $\|y\| = \sqrt{y^\top P y}$ with P satisfying (3.10).

Let the gain vector K and a positive definite matrix $P \succ 0$ be selected as follows:

$$(A_0 + BK + G_{\mathbf{d}})^\top P + P(A_0 + BK + G_{\mathbf{d}}) = 0, \quad PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \tag{6.36}$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is the generator of the dilation $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}, s \in \mathbb{R}$. Such a selection is always possible (see Polyakov *et al.*, 2016b or Polyakov, 2020). In this case, we derive

$$\tilde{f}(y) := \left(\frac{(I_n - G_{\mathbf{d}}) y y^\top P}{y^\top P G_{\mathbf{d}} y} + I_n \right) (A + BK) \frac{y}{\|y\|} = \frac{1}{\|y\|} \frac{(I_n - G_{\mathbf{d}}) y y^\top P (A + BK) y}{y^\top P G_{\mathbf{d}} y} + (A + BK) \frac{y}{\|y\|} =$$

$$\frac{1}{\|y\|} \frac{(I_n - G_{\mathbf{d}}) y (-y^\top P G_{\mathbf{d}} y)}{y^\top P G_{\mathbf{d}} y} + (A + BK) \frac{y}{\|y\|} = (A + BK + G_{\mathbf{d}} - I_n) \frac{y}{\|y\|}.$$

Case $\nu = -1$.

For the homogeneous system with negative degree we apply Theorem 6.9. The consistent discretization (6.20) has the following representation

$$y_i \in y_{i+1} + h \left(I_n - \tilde{A} \right) \tilde{F}(y_{i+1}), \quad h > 0, \ i = 0, 1, 2, \ldots \tag{6.37}$$

where $\tilde{A} = A + BK + G_{\mathbf{d}}$ such that $\tilde{A}^\top P + P\tilde{A} = 0$ and

$$\tilde{F}(y) = \begin{cases} \left\{ \frac{y}{\|y\|} \right\} & \text{if} \quad y \neq 0 \\ B(1) & \text{if} \quad y = 0, \end{cases}$$

where $B(1)$ is the unit ball in $\mathbb{R}^n$ with the norm $\|y\| = \sqrt{y^\top P y}$. Notice that the condition (6.36) implies that $I_n - \tilde{A}$ is invertible.

Let us denote $q_{i+1} = \|y_{i+1}\|$ and $z_{i+1} = \frac{y_{i+1}}{\|y_{i+1}\|}$. Then the inclusion (6.37) has the following solution

- if $y_i^\top (I_n - \tilde{A})^{-\top} P (I_n - \tilde{A})^{-1} y_i \le h^2$ then
$$q_{i+1} = 0 \quad \text{and} \quad z_{i+1} = h^{-1}(I_n - h\tilde{A})^{-1} y_i; \tag{6.38}$$
- otherwise, q_{i+1} and z_{i+1} are derived as the solution to
$$\left((q_{i+1}+h)I_n - h\tilde{A}\right) z_{i+1} = y_i, \quad z_{i+1}^\top P z_{i+1} = 1, \tag{6.39}$$

where $y_i = \|x_i\|_{\mathbf{d}} \mathbf{d}(-\ln \|x_i\|_{\mathbf{d}}) x_i$. Solution to (6.39) always exists due to Theorem 6.9. To find it the equation

$$y_i^\top \left((q_{i+1}+h)I_n - h\tilde{A}\right)^{-\top} P \left((q_{i+1}+h)I_n - h\tilde{A}\right)^{-1} y_i = 1,$$

that is polynomial with respect to q_{i+1}, must be initially solved. For $n = 2$ the system (6.39) implies a quartic equation with respect to q_{i+1}, so it can be solved explicitly using Ferrari formulas. In other cases some proper computational procedure can be utilized. In all numerical experiments we consider the model of the controlled double integrator:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The simulation results for $\nu = -1$, $x_0 = \begin{pmatrix} 0.2247 & 0.4494 \end{pmatrix}^\top$ and

$$\mathbf{d}(s) = \begin{pmatrix} e^{2s} & 0 \\ 0 & e^{s} \end{pmatrix}, \quad P = \begin{pmatrix} 9.1050 & 1.7829 \\ 1.7829 & 0.8914 \end{pmatrix},$$

$$K = \begin{pmatrix} -10.2139 & -3.0000 \end{pmatrix},$$

are given in Fig. 6.1, where the developed discretization scheme is compared with the explicit Euler scheme. The simulations confirm finite-time convergence of $\{x_i\}$ to zero in a finite number of steps for the consistent discrete-time model, where

$$x_i = \mathbf{d}(\ln \|y_i\|) \frac{y_i}{\|y_i\|} \tag{6.40}$$

and $\{y_i\}$ is the solution to (6.37), while the system obtained using the explicit Euler discretization is not even asymptotically stable (see Fig. 6.1).

Case $\nu = 1$.

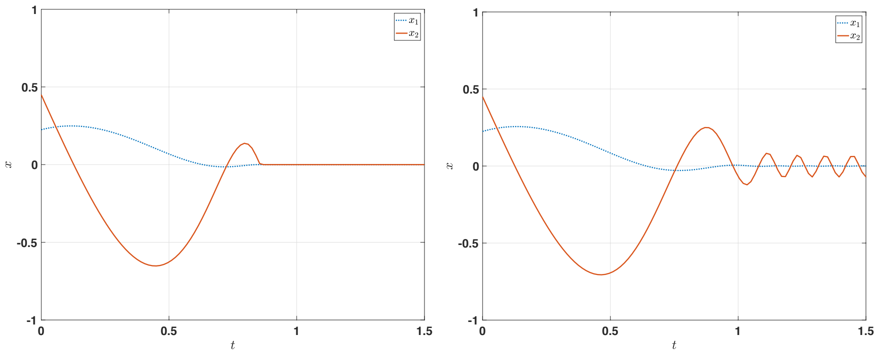

Figure 6.1: Comparison of the discrete-time models obtained by means of the consistent discretization (left) and explicit Euler method (right) for $h = 0.015$ and $\nu = -1$.

In this case we use Theorem 6.11. The semi-implicit discretization (6.23) gives

$$\left(I_n + h\|y_i\|(I_n - \tilde{A})\right) y_{i+1} = y_i, \quad i = 0, 1, 2, \ldots$$

The matrix $\left(I_n + h\|y_i\|(I_n - \tilde{A})\right)$ has only positive eigenvalues for any $h > 0$ and any $\|y_i\|$ (since $\tilde{A}^\top P + P\tilde{A} = 0$), then it is invertible, so

$$y_{i+1} = \left(I_n + h\|y_i\|(I_n - \tilde{A})\right)^{-1} y_i. \tag{6.41}$$

The results of the numerical simulation for $\nu = 1$, $n = 2$, $x_0 = \begin{pmatrix} 13.255 & 0 \end{pmatrix}^\top$ and

$$\mathbf{d}(s) = \begin{pmatrix} e^s & 0 \\ 0 & e^{2s} \end{pmatrix}, \quad P = \begin{pmatrix} 3.6173 & 2.6173 \\ 2.6173 & 2.6173 \end{pmatrix},$$

$$K = \begin{pmatrix} -1.3821 & -3.0000 \end{pmatrix}$$

are presented in Fig. 6.2, where the developed consistent discretization scheme is compared with the explicit Euler scheme.

The simulations show an oscillatory behavior of the discrete-time model obtained using the explicit Euler scheme for $h = 0.04$ (see Fig. 6.2). For $h > 0.05$ the explicit scheme was found to be unstable (solution blows up for the given x_0).

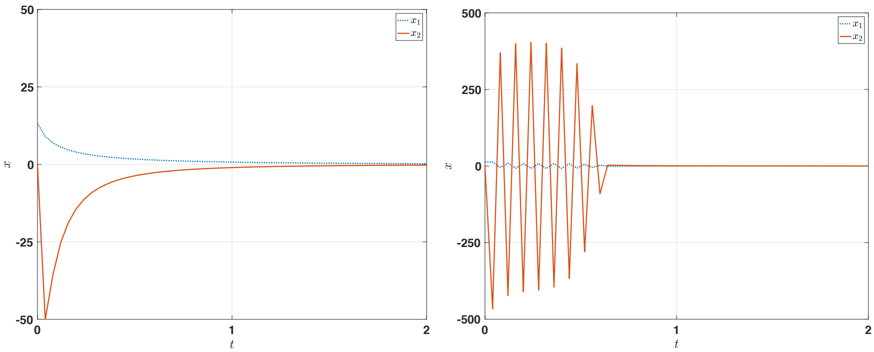

Figure 6.2: Comparison of the discrete-time models obtained by means of the consistent discretization (left) and explicit Euler method (right) for $h = 0.04$ and $\nu = 1$.

The consistency of the discrete-time model (6.41), (6.40), obtained using the discretization scheme (6.23), is confirmed by numerical experiments. The nearly fixed-time stability is observed in simulations even for large sampling periods ($h = 1$) and large initial conditions ($\|x_0\|$ is of the order 10^{20}).

On digital implementation of a homogeneous control using the consistent discretization

The discretization schemes given above are developed for a numerical simulation of finite-time and fixed-time stable homogeneous ODEs. However, they can also be utilized for a digital (sampled-time) implementation of finite-time or fixed-time controllers. Indeed, let us consider again the example given in the previous section (Case $\nu = -1$).

The control (6.35) for $\nu = -1$ is proven to be robust with respect to bounded disturbances (see Corollary 4.6), i.e. the origin of the continuous-time closed-loop system

$$\dot{x} = Ax + B(u_\nu(x) + \gamma(t, x)), \quad t > 0 \tag{6.42}$$

remains globally uniformly finite-time stable provided that $|\gamma| < \gamma_0$ and γ_0 is sufficiently small.

Taking into account $x = \mathbf{d}(\ln \|y\|)\frac{y}{\|y\|}$ and $u_\nu(\mathbf{d}(s)x) = u_\nu(x)$ for $\nu = -1$, we derive that the consistent discretization of the control law (6.35) can be defined as follows

$$u_\nu(x_{i+1}) = u_\nu\left(\frac{y_{i+1}}{\|y_{i+1}\|}\right) = u_\nu(z_{i+1}) = Kz_{i+1},$$

where z_{i+1} is given by (6.38) or (6.39). According to the conventional implicit discretization technique (Huber *et al.*, 2016; Acary *et al.*, 2012; Miranda-Villatoro *et al.*, 2017; Miranda-Villatoro *et al.*, 2018), this value is suggested to be selected for the time interval $[t_i, t_{i+1})$ during a digital implementation of the control law (6.35) in the system (6.34):

$$u(t) = u_i := Kz_{i+1}, \qquad t \in [ih, (i+1)h). \tag{6.43}$$

The solution to the perturbed system (6.42) in this case is given by

$$x(t) = e^{At}x_i + \int_0^t e^{A(t-s)}B(u_i + \gamma(s, x(s))\,ds, t \in [ih, (i+1)h), x_i := x(ih). \tag{6.44}$$

Case $\gamma \equiv 0$. The implicit sampled control (6.43) obtained using the consistent discretization completely rejects the numerical chattering (the numerical chattering is unmodeled oscillations in a control system caused by discretization errors of a continuous-time control algorithm, (see Huber *et al.*, 2016 for more details and Figure 6.4) in both input and state of the system, while the explicit scheme $u_i = u_\nu(x_i)$ always generates the chattering (see Figure 6.3). The simulation results for $\gamma = 0$ and another sampling periods can be also found in Polyakov *et al.* (2019).

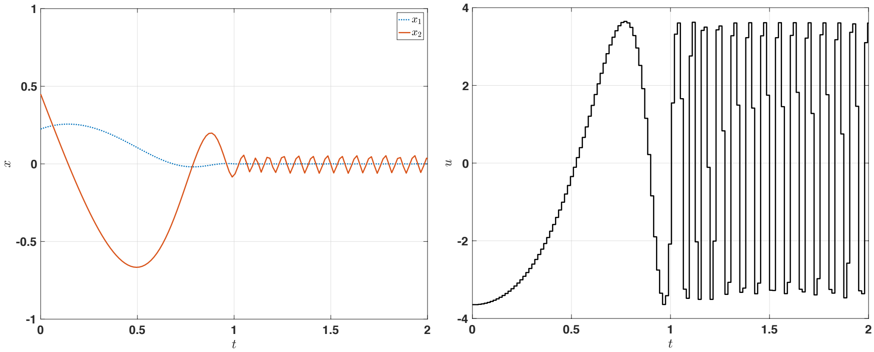

Figure 6.3: Evolution of the system (6.44) with the explicitly discretized control: $u_i = u_\nu(x_i)$, $h = 0.015$, $\nu = -1$ and $\gamma = 0$.

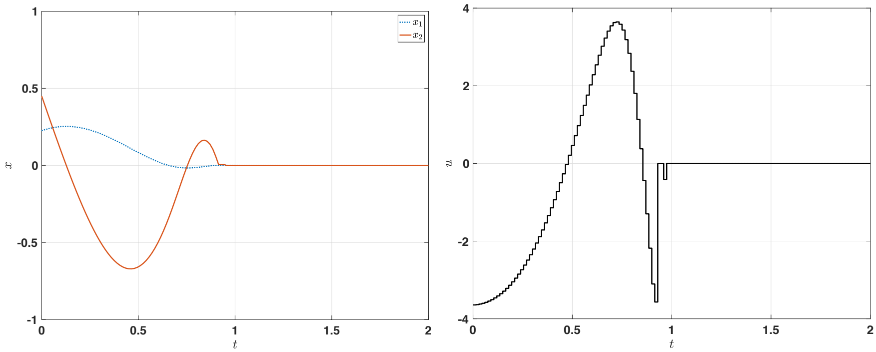

Figure 6.4: Evolution of the system (6.44) with the consistently discretized control: $u_i = Kz_{i+1}$, $h = 0.015$, $\nu = -1$ and $\gamma = 0$.

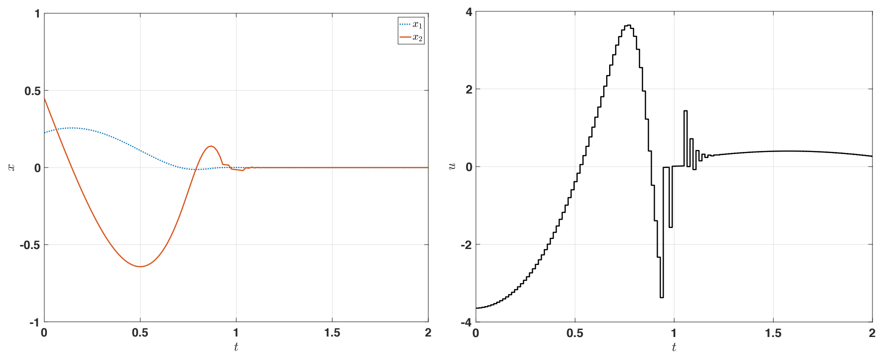

Figure 6.5: Evolution of the system (6.44) with the consistently discretized control $u_i = Kz_{i+1}$, $h = 0.015$ and $\gamma = 0.4\cos(2t)$.

Case $\gamma \neq 0$. The implicit homogeneous control remains efficient for rejection of the perturbation $\gamma = 0.4\cos(2t)$ (see Figure 6.5). Noised measurements imply an expectable degradation of the control precision (Figure 6.6). However, the chattering magnitude of the consistently discretized controller in the noised and perturbed case is still less than the chattering magnitude of the explicitly discretized controller in the disturbance-free case.

Notice that the consistently discretized implicit homogeneous control tracks (rejects) the matched perturbation $\gamma = 0.4\cos(2t)$ since $u \approx -\gamma$ when $x \approx \mathbf{0}$ (see the right figure 6.5). This happens without any knowledge about γ. Such a behavior is observed only for "slowly-varying"

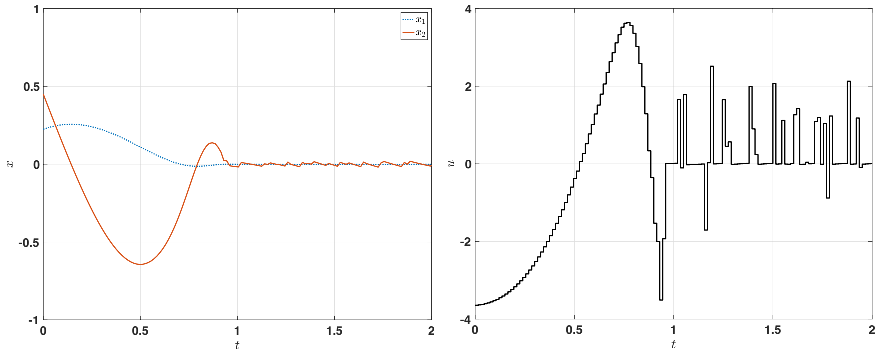

Figure 6.6: Evolution of the system (6.44) with the consistently discretized control $u_i = Kz_{i+1}$, $h = 0.015$, $\gamma = 0.4\cos(2t)$ and with a uniformly distributed measurement noise of the magnitude 10^{-3}

perturbations. To reject a "faster" disturbance the sampling period has to be decreased. In the view of Corollary 4.6, a set of perturbations to be rejected should "tend" to a ball in L^∞ as $h \to 0$.

7

Extensions

7.1 Time-delay systems

The definitions of FTS/FxTS can be extended to time-delay systems with mild modifications. The differences appear in our methods to verify these stability properties. For ODEs, two methods can be applied based on Lyapunov functions and the theory of homogeneous systems.

The homogeneity concept can also be adapted to time-delay systems, either using the fact that any retarded system can be embedded in an evolution equation (Polyakov *et al.*, 2016a) or developing directly the notion of weighted homogeneity to this kind of dynamics (Efimov *et al.*, 2014a; Efimov *et al.*, 2016). In the former case, all results previously obtained for ODE systems can be recovered in the time-delay scenario, however, at the price that in such a homogeneous system the delay has to be scaled with the state. The latter approach leads to the solutions admitting a constant delay, but the stability of these homogeneous systems with negative/positive degree does not imply FTS/FxTS (Efimov *et al.*, 2016) (some additional restrictions related with uniformity of convergence have to be introduced).

Returning to Lyapunov's ideas, there exist two generic frameworks assessing stability of time-delay systems, which are based on analysis of a Lyapunov-Razumikhin function or a Lyapunov-Krasovskii functional (Kolmanovskii and Myshkis, 1999; Fridman, 2014). The latter method has been proven to be equivalent to the asymptotic stability property for some particular classes of the time-delay systems (Pepe and Karafyllis, 2013; Pepe *et al.*, 2017; Efimov and Fridman, 2020), and it can also be used to establish finite-time stability (Moulay *et al.*, 2008). The former approach is only sufficient for the asymptotic stability (Kolmanovskii and Myshkis, 1999; Fridman, 2014), and it is less intuitive while obtaining the rate of solution convergence (Myshkis, 1995; Aleksandrov and Zhabko, 2012; Efimov *et al.*, 2014b; Efimov and Aleksandrov, 2020). An advantage of Lyapunov-Razumikhin approach with respect to Lyapunov-Krasovskii one is that in many nonlinear cases it is simpler to find a Lyapunov-Razumikhin function than a Lyapunov-Krasovskii functional (Efimov *et al.*, 2014a; Efimov *et al.*, 2016) (*e.g.*, a Lyapunov function for the delay-free case can be tested).

In this subsection, the definitions of accelerated convergence rates are described for time-delay systems. Next, the necessary and sufficient conditions are formulated using Lyapunov-Krasovskii approach. The Lyapunov-Razumikhin approach is finally recalled.

7.1.1 Preliminaries

We denote by $C_{[a,b]} = C([a,b],\mathbb{R}^n)$, $-\infty < a < b < +\infty$ the Banach space of continuous functions $\phi : [a,b] \to \mathbb{R}^n$ with the uniform norm $\|\phi\| = \sup_{a\leq\varsigma\leq b} \|\phi(\varsigma)\|_{\mathbb{R}^n}$.

Consider an autonomous functional differential equation of retarded type (Kolmanovsky and Nosov, 1986):

$$\frac{dx(t)}{dt} = f(x_t),\ t \geq 0, \tag{7.1}$$

where $x(t) \in \mathbb{R}^n$ and $x_t \in C_{[-\tau,0]}$ is the state function, $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$ and $\tau > 0$ is a finite delay; $f \in C(C_{[-\tau,0]},\mathbb{R}^n)$ with $f(\mathbf{0}) = 0$ and it is such that the solutions in forward time for the system (7.1) exist and are unique (Kolmanovsky and Nosov, 1986). Let $x(t,x_0)$ denote such a unique solution satisfying the initial condition $x(s,x_0) = x_0(s)$ for $-\tau \leq s \leq 0$ and $x_0 \in C_{[-\tau,0]}$, which is defined on some finite time interval $[-\tau,T)$ with $0 < T \leq +\infty$.

For a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ the upper directional Dini derivative is defined as follows:

$$D^+V(x)v = \limsup_{h\to 0^+} \frac{V\left(x+hv\right)-V\left(x\right)}{h}$$

for any $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$. The upper right-hand Dini derivative of a continuous functional $V : C_{[-\tau,0]} \to \mathbb{R}$ along the system (7.1) solutions is defined as

$$D^+V(\phi) = \limsup_{h\to 0^+} \frac{V(\phi_h)-V(\phi)}{h}$$

for any $\phi \in C_{[-\tau,0]}$, where $\phi_h \in C_{[-\tau,0]}$ for $0 < h < \tau$ is given by

$$\phi_h = \begin{cases} \phi(\theta+h), & \theta \in [-\tau,-h) \\ \phi(0)+f(\phi)(\theta+h), & \theta \in [-h,0]. \end{cases}$$

7.1.2 Stability definitions

Let Ω be a neighborhood of zero in $C_{[-\tau,0]}$.

Definition 7.1. (Moulay *et al.*, 2008; Kolmanovskii and Myshkis, 1999; Fridman, 2014; Efimov and Aleksandrov, 2020) The origin of the system (7.1) is said to be

(*a*) *stable* if there is $\sigma \in \mathcal{K}$ such that for any $x_0 \in \Omega$, the solutions are defined and $\|x(t,x_0)\| \leq \sigma(\|x_0\|)$ for all $t \geq 0$;

(*b*) *asymptotically stable* if it is stable and $\lim_{t\to+\infty} \|x(t,x_0)\| = 0$ for any $x_0 \in \Omega$;

(*c*) *FTS* if it is stable and for any $x_0 \in \Omega$ there exists $0 \leq T^{x_0} < +\infty$ such that $x(t,x_0) = 0$ for all $t \geq T^{x_0}$. The functional $T(x_0) = \inf\{T^{x_0} \geq 0 : x(t,x_0) = 0\ \forall t \geq T^{x_0}\}$ defines the settling time of the system (7.1);

(*d*) *nearly FxTS* if it is stable and for any $\varrho > 0$ there exists $0 < T_\varrho < +\infty$ such that $\|x(t,x_0)\| \leq \varrho$ for all $t \geq T_\varrho$ and all $x_0 \in \Omega$;

(*e*) *FxTS* if it is FTS and $\sup_{x_0\in\Omega} T(x_0) < +\infty$.

If $\Omega = C_{[-\tau,0]}$, then the corresponding properties are called *global*.

Remark 7.1. The definition of global asymptotic stability can be given in terms of existence of a function $\beta \in \mathcal{KL}$ such that $\|x(t,x_0)\| \leq \beta(\|x_0\|,t)$

for all $x_0 \in C_{[-\tau,0]}$ and $t \geq 0$. In such a case, if $\beta(s,t) = cse^{-at}$ for some $c \geq 1$ and $a > 0$, then the system (7.1) is called globally *exponentially stable*.

Similarly as in Section 2, the property of nearly FxTS is conceptually different from FxTS notion, since in the former case $\sup_{\varrho>0} T_\varrho \leq +\infty$, *i.e.*, at the origin a nearly FxTS system (7.1) may be just asymptotically stable. On other words, a FxTS system has to be FTS and nearly FxTS at the origin simultaneously. The difference between nearly FxTS and asymptotically stable systems becomes important for an unbounded set Ω only.

Note that due to uniqueness of solutions of (7.1), for any $x_0 \in \Omega$ and $t \in \mathbb{R}_+$:

$$T(x_t) = \max\{T(x_0) - t, 0\}, \tag{7.2}$$

which can be shown by repeating the arguments of Bhat and Bernstein (2000).

In Efimov *et al.* (2014b) a necessary condition of finite-time stability of (7.1) is discussed, which is based on the observation that

$$x(t,x_0) = 0 \quad \forall t \geq T(x_0) \Rightarrow f(x_t) = 0 \quad \forall t \geq T(x_0),$$

but $x_{T(x_0)} \neq \mathbf{0}$ according to Definition 7.1 (we need also continuity of f here). Recall that initially it has been assumed only $f(\mathbf{0}) = 0$, but the observation above necessary leads to a more strong restriction: there exists a nonempty set $\Omega_0 \subseteq \Omega \cap C^0_{[-\tau,0]}$, where

$$C^0_{[-\tau,0]} = \{\phi \in C_{[-\tau,0]} : \phi(0) = 0\},$$

such that

$$f(\phi) = 0 \quad \forall \phi \in \Omega_0. \tag{7.3}$$

If we will consider the limit case with

$$\Omega_0 = \Omega \cap C^0_{[-\tau,0]},$$

which is assumed in the sequel, then several results discovered in FTS systems described by ODEs can be recovered for (7.1). Note that in such a case we have

$$T(x_0) = 0 \quad \forall x_0 \in \Omega_0$$

provided that solutions are unique in the forward time.

Proposition 7.1. If solutions of the FTS system (7.1) depend continuously of initial conditions uniformly on compact intervals of time, *i.e.*, for any finite $\delta > 0$

$$\sup_{t\in[0,\delta]} \|x(t,x^*) - x(t,x_0)\| \to 0 \text{ as } x^* \to x_0 \in \Omega\backslash\Omega_0$$

then $T \in C(\Omega, \mathbb{R}_+)$ provided that T is continuous at any point of the set Ω_0.

Proof. Take $x_0 \in \Omega\backslash\Omega_0$ and an arbitrary sequence $x_i \in \Omega$, $i \in \mathbb{N}$ converging to x_0. For the solutions $x(t, x_i)$, denote by x_t^i the state of the system (7.1) at time instant t with the initial condition x_i ($x_t = x_t^0$ as before).

On the one hand, from the definition of the settling-time functional we derive that $x_{T(x_0)} \in \Omega_0$. On the other hand, the continuous dependence of solutions on initial conditions implies

$$\lim_{i\to+\infty} x(T(x_0), x_i) = x(T(x_0), x_0) = 0 \text{ and } \lim_{i\to+\infty} \|x^i_{T(x_0)} - x_{T(x_0)}\| = 0.$$

Hence, the continuity of T at $x_{T(x_0)}$ implies

$$\lim_{i\to+\infty} T(x^i_{T(x_0)}) = T(x_{T(x_0)}) = 0.$$

Then due to (7.2) we have

$$0 = \lim_{i\to+\infty} T(x^i_{T(x_0)}) = \lim_{i\to+\infty} \max\{T(x_i) - T(x_0), 0\},$$

i.e., $\limsup_{i\to+\infty} T(x_i) \le T(x_0)$.

Let $T^- = \liminf_{i\to+\infty} T(x_i)$. Then there exists a sub-sequence x_{i_j} such that $\lim_{j\to+\infty} T(x_{i_j}) = T^-$. Since due to continuous dependence of solutions on initial conditions and on the time argument we have $|x(T(x_{i_j}), x_{i_j}) - x(T^-, x_0)| \le |x(T(x_{i_j}), x_{i_j}) - x(T^-, x_{i,j})| + |x(T^-, x_{i_j}) - x(T^-, x_0)| \to 0$ as $j \to +\infty$ then

$$0 = \lim_{j\to+\infty} x(T(x_{i_j}), x_{i_j}) = x(T^-, x_0).$$

The latter means $x_{T^-} \in \Omega_0$ and $T^- \ge T(x_0)$ by the definition of T, *i.e.*, $\liminf_{i\to+\infty} T(x_i) = T^- \ge T(x_0)$.

Combining the estimates for lim inf and lim sup we conclude $T(x_i) \to T(x_0)$ as $i \to +\infty$. □

Note that the requirement on continuity of T at any point of the set Ω_0 imposed in the last proposition means a local stability of the set Ω_0 for the solutions of (7.1). Indeed, in such a case the properties of T imply existence of some $\varrho_1, \varrho_2 \in \mathcal{K}_\infty$ such that $\varrho_1(\|x\|_{\Omega_0}) \leq T(x) \leq \varrho_2(\|x\|_{\Omega_0})$ for all x in a vicinity $\mathcal{O}$ of Ω_0, where $\|x\|_{\Omega_0} = \inf_{\phi\in\Omega_0} \|x - \phi\|$. Hence, take any $x_0 \in \mathcal{O}$, from (7.2) we have:

$$\varrho_1(\|x_t\|_{\Omega_0}) \leq T(x_t) \leq T(x_0) \leq \varrho_2(\|x_0\|_{\Omega_0})$$

for all $t \in [0, T(x_0))$, which implies the stability of Ω_0 (Kolmanovsky and Nosov, 1986).

Corollary 7.1. Let f in the FTS system (7.1) satisfy the property (7.3) and for any bounded set $X \in \Omega \setminus \Omega_0$, there exist $L > 0$ such that $\|f(\phi) - f(\varphi)\| \leq L\|\phi - \varphi\|$ for all $\phi, \varphi \in X$, then $T \in C(\Omega, \mathbb{R}_+)$ provided that T is continuous at any point of the set Ω_0.

Proof. It is enough to show that under introduced hypotheses the solutions of (7.1) depend continuously in initial conditions uniformly on compact intervals of time $t \geq 0$. Integrating (7.1) for any $x_1, x_2 \in \Omega\setminus\Omega_0$ on the interval $t \in [0, T_{12})$, where $T_{12} = \min\{T(x_1), T(x_2)\}$, for the respective solutions $x_t^{x_1}, x_t^{x_2}$ we obtain:

$$x(t, x_i) = x_i(0) + \int_0^t f(x_s^{x_i})ds, \ i = 1, 2.$$

Since the solutions $x_t^{x_1}, x_t^{x_2}$ belong to a bounded set $\mathcal{X} \subset \Omega$ due to the FTS, and using the Lipschitz property of f, there exists $L > 0$ on $\mathcal{X} \setminus \Omega_0$ such that

$$\begin{aligned}\|x_t^{x_1} - x_t^{x_2}\| &\leq \|x_1 - x_2\| + \int_0^s \|f(x_s^{x_1}) - f(x_s^{x_2})\|ds \\ &\leq \|x_1 - x_2\| + L\int_0^t \|x_s^{x_1} - x_s^{x_2}\|ds\end{aligned}$$

for any $t \in [0, T_{12})$. Denote

$$m(t) = \|x_1 - x_2\| + L\int_0^t \|x_s^{x_1} - x_s^{x_2}\|ds,$$

then

$$\frac{dm(t)}{dt} = L\|x_t^1 - x_t^2\| \leq Lm(t),$$

from which we derive

$$\|x_t^{x_1} - x_t^{x_2}\| \leq m(t) \leq e^{Lt} m(0) = e^{Lt}\|x_1 - x_2\|$$

for all $t \in [0, T_{12})$, which implies the desired continuity in initial conditions uniformly on $[0, T_{12}]$. For $t \in [T_{12}, T^{12})$, where $T^{12} = \max\{T(x_1), T(x_2)\}$, we get:

$$\begin{aligned}\|x_t^{x_{T_{12}}^{x_1}} - x_t^{x_{T_{12}}^{x_2}}\| &\leq \|x_{T_{12}}^{x_1} - x_{T_{12}}^{x_2}\| + \int_{T_{12}}^{T^{12}} \|f(x_s^{x_{T_{12}}^{x_1}}) - f(x_s^{x_{T_{12}}^{x_2}})\| ds \\ &\leq \|x_{T_{12}}^{x_1} - x_{T_{12}}^{x_2}\| + \int_{T_{12}}^{T^{12}} M ds,\end{aligned}$$

where $M = \sup_{t \in [T_{12}, T^{12})} \|f(x_t^{x_{T_{12}}^{x_1}}) - f(x_t^{x_{T_{12}}^{x_2}})\|$. Note that T_{12}, T^{12} and M admit uniform upper bounds on $\mathcal{X}$. Finally,

$$\begin{aligned}\|x_t^{x_{T_{12}}^{x_1}} - x_t^{x_{T_{12}}^{x_2}}\| &\leq \|x_{T_{12}}^{x_1} - x_{T_{12}}^{x_2}\| + M|T(x_1) - T(x_2)| \\ &\leq e^{LT_{12}}\|x_1 - x_2\| + M \max\{T(x_{T_{12}}^{x_1}), T(x_{T_{12}}^{x_2})\},\end{aligned}$$

and recall that $\inf_{\phi \in \Omega_0} \|x_{T_{12}}^{x_i} - \phi\| \to 0$ as $x_1 \to x_2$ for $i = 1, 2$, *i.e.*, $x_{T_{12}}^{x_i}$ approaching Ω_0 if the distance between x_1 and x_2 is decreasing, then $\max\{T(x_{T_{12}}^{x_1}), T(x_{T_{12}}^{x_2})\} \to 0$ as $x_1 \to x_2$ due to assumed continuity of T on Ω_0. Obviously,

$$\|x_t^{x_{T^{12}}^{x_1}} - x_t^{x_{T^{12}}^{x_2}}\| = 0$$

for all $t \geq T^{12}$, and the required continuity of solutions of (7.1) in the initial conditions uniformly on compact intervals of time is established.

Next, the proof repeats the substantiation of Proposition 7.1. □

It has been also observed that for time-delay systems the convergence rates, which are usual for ODEs, can be rare, while a more natural accelerated convergence is hyperexponential (Polyakov *et al.*, 2015b) (that is faster than any exponential estimate).

7.1.3 Application of Lyapunov-Krasovskii approach

For introduced stability notions of (7.1) there exist the following sufficient conditions:

Proposition 7.2. Let the system (7.1) admit $V \in C(\Omega, \mathbb{R}_+)$, $\eta_1, \eta_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$ and $\epsilon > 0$ such that $\dot{z}(t) = -\rho(z(t))$ has a flow for all $z(0) \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$, with

$$\eta_1(\|\phi(0)\|) \leq V(\phi) \leq \eta_2(\|\phi\|),\ D^+V(\phi) \leq -\rho(V(\phi))$$

for all $\phi \in \Omega$.

If

$$\int_0^\epsilon \frac{dz}{\rho(z)} < +\infty,$$

then the system (7.1) is FTS at the origin with the settling time satisfying an upper estimate:

$$T(\phi) \leq \int_0^{V(\phi)} \frac{dz}{\rho(z)}.$$

If

$$\int_\epsilon^{\sup_{\phi \in \Omega} V(\phi)} \frac{dz}{\rho(z)} < +\infty,$$

then the system (7.1) is nearly FxTS, and it is FxTS at the origin if $\epsilon = 0$.

Proof. The FTS and nearly FxTS cases were proven in Moulay *et al.* (2008) and Efimov and Aleksandrov (2020), respectively. The FxTS result follows from their combination. □

A usual example of such a function ρ providing FTS property includes

$$\rho(z) = az^\alpha$$

for $a > 0$ and $\alpha \in [0, 1)$. Then it is straightforward to check that under the conditions of Proposition 7.2, the solutions of system (7.1) admit an upper estimate for $x_0 \in \Omega$ and all $t \geq 0$:

$$\|x(t, x_0)\| \leq \max\{0, \eta_1^{-1} \circ \left(\eta_2^{1-\alpha}(\|x_0\|) - a(1-\alpha)t\right)^{\frac{1}{1-\alpha}}\},$$

hence, the system is FTS with

$$T(x_0) \leq \frac{\eta_2^{1-\alpha}(\|x_0\|)}{a(1-\alpha)}.$$

If $\alpha = 1$, then the system is asymptotically stable:

$$\|x(t, x_0)\| \le \eta_1^{-1}\left(\eta_2(\|x_0\|)\exp(-at)\right)$$

for all $t \ge 0$ and $x_0 \in \Omega$. Finally, if $\alpha > 1$, then (7.1) is nearly FxTS:

$$\|x(t, x_0)\| \le \eta_1^{-1}\left(\frac{1}{\left(\eta_2^{1-\alpha}(\|x_0\|) + a(\alpha-1)t\right)^{\frac{1}{\alpha-1}}}\right)$$

for all $t \ge 0$ and $x_0 \in \Omega$. A choice

$$\rho(z) = a(z^\alpha + z^\beta)$$

with $a > 0$, $\alpha \in [0, 1)$ and $\beta > 1$ is conventional for FxTS as before.

Note that comparing with the results for ODEs, the conditions of Proposition 7.2 do not imply continuity of the settling-time function (since there is no statement that continuity at the origin implies this property globally for T), however recalling the conditions of Proposition 7.1 that results can be recovered for time-delay systems:

Theorem 7.2. Let solutions of the system (7.1) depend continuously of initial conditions uniformly on compact intervals of time, then the origin of (7.1) is FTS with a continuous settling-time functional T if and only if there exist $V \in C(\Omega, \mathbb{R}_+)$, $\eta_1, \eta_2 \in \mathcal{K}_\infty$, $c > 0$ and $\alpha \in (0, 1)$ such that

$$\eta_1(\|\phi(0)\|) \le V(\phi) \le \eta_2(\|\phi\|),\ D^+V(\phi) \le -cV^\alpha(\phi)$$

for all $\phi \in \Omega$.

Proof. For the FTS case, existence of such a Lyapunov-Krasovskii functional V implies this stability property (V is positive definite with respect to the set Ω^0) and continuity of T follows from the inequality $T(\phi) \le \frac{V^{1-\alpha}(\phi)}{(1-\alpha)c}$ for all $\phi \in \Omega$. Conversely, if the system is FTS, take $V(\phi) = \frac{T^{1+\gamma}(\phi)}{1+\gamma}$ for some $\gamma > 0$, then $\eta_1(s) = \inf_{\|\phi(0)\|=s} V(\phi)$, $\eta_2(s) = \sup_{\|\phi\| \le s} V(\phi)$ and

$$D^+V(\phi) = T^\gamma(\phi)D^+T(\phi)$$

for any $\phi \in \Omega$. Recall (7.2), then $D^+T(\phi) = -1$ for $\phi \in \Omega \setminus \Omega^0$ and $D^+T(\phi) = 0$ for $\phi \in \Omega^0$, hence

$$D^+V(\phi) \leq -cV^\alpha(\phi)$$

for all $\phi \in \Omega$, where $c = (1+\gamma)^\alpha$ and $\alpha = \frac{\gamma}{1+\gamma} \in (0,1)$ as needed. □

Example 7.1. Let us show that the restriction $\Omega_0 = \Omega \cap C^0_{[-\tau,0]}$ used in Proposition 7.1 and Theorem 7.2 is indeed important. Consider

$$\dot{x}(t) = -2x^{\frac{1}{3}}(t) - \left(\max_{\theta\in[-\tau,0]} x(t+\theta)\right)^{\frac{1}{3}}, \tag{7.4}$$

where $x(t) \in \mathbb{R}$. It is easy to check that the system is globally asymptotically stable using the Lyapunov-Razumikhin function $V(x) = \|x\|$. Moreover, it is also globally FTS. Indeed, if $x(\theta) \leq 0$ for all $\theta \in [-\tau, 0]$, then at the instant $t' \geq 0$ such that $x(t') = 0$ for the first time (such an instant exists since $\dot{V}(t) \leq -2V^{\frac{1}{3}}(t)$ for all $t \in [0, t']$) we have that $\max_{\theta\in[-\tau,0]} x(t'+\theta) = 0$ (*i.e.*, $\dot{x}(t') = 0$) and, hence, $\max_{\theta\in[-\tau,0]} x(t+\theta) = 0$ for all $t \in [t', t'+\tau]$ and actually $T(x_0) = t'$. If $\max_{\theta\in[-\tau,0]} x(\theta) > 0$, then there exists $t' \geq 0$ such that $x(t') = 0$ for the first time (such an instant exists by the same argument), $\dot{x}(t') < 0$ due to the equation of the system, and $x(t) \leq 0$ for all $t \geq t'$, which implies next the finite-time convergence by the reasoning given for the previous case.

Note that this system is continuous and homogeneous of a negative degree in the sense of Efimov *et al.* (2014a), the property (7.3) holds for $\Omega_0 = \{\phi \in C^0_{[-\tau,0]} : \phi(\theta) \leq 0,\ \theta \in [-\tau, 0]\} \subset \Omega \cap C^0_{[-\tau,0]}$ (the conditions of Proposition 7.1 are not satisfied), and the settling-time functional is discontinuous at Ω_0. To check this we can consider any arbitrary small $x_0 \in \Omega \setminus \Omega^0$ with $x(\theta) > 0$ for all $\theta \in [-\tau, 0]$ whose settling time cannot be smaller than τ (the trajectory $x(t)$ has to become negative for all $[t-\tau, t]$ before a convergence till the origin can emerge).

An extension of the converse results for (nearly) FxTS requires additional investigations, since an application of the arguments used in the ODE case requires the concept of solutions in the inverse time to be defined, which is not developed for time-delay systems.

7.1.4 Application of Lyapunov-Razumikhin approach

For the Lyapunov-Krasovskii approach there are converse results for asymptotic stability (Pepe and Karafyllis, 2013; Pepe *et al.*, 2017; Efimov and Fridman, 2020), while the Lyapunov-Razumikhin method is only sufficient (Kolmanovsky and Nosov, 1986). In addition, evaluations by the Lyapunov-Razumikhin approach of the convergence rates requires substantial modifications of its formulation:

Proposition 7.3. (Myshkis, 1995) Let there exist a locally Lipschitz continuous Lyapunov-Razumikhin function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that

(i) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and all $x \in \mathbb{R}^n$:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|); \tag{7.5}$$

(ii) for some $\gamma' > 1$, $\alpha' > 0$ and all $\varphi \in C_{[-\tau,0]}$:

$$\max_{\theta\in[-\tau,0]} V\left(\varphi(\theta)\right) \leq \gamma' V\left(\varphi(0)\right) \Rightarrow$$
$$D^+ V\left(\varphi(0)\right) f(\varphi) \leq -\alpha' V\left(\varphi(0)\right).$$

Then the origin is globally asymptotically stable for the system (7.1) with exponential rate of convergence, and for all $x_0 \in C_{[-\tau,0]}$ and $t \geq 0$:

$$|x(t, x_0)| \leq \alpha_1^{-1}\left(\exp\left(-\min\{\alpha', \frac{\ln\gamma'}{\tau}\}t\right)\alpha_2\left(\|x_0\|\right)\right).$$

A similar estimate can be derived from Halanay's inequality (Halanay, 1966).

The above result is given to recall the formulation of the Lyapunov-Razumikhin approach: if we replace constants γ', α' with function $\gamma', \alpha' \in \mathcal{K}$ of V, such that $\gamma'(s) > s$ for all $s > 0$, then the conventional sufficient Lyapunov-Razumikhin condition for asymptotic stability of (7.1) is recovered (Kolmanovsky and Nosov, 1986). The formulations for accelerated convergence rates are more complex:

Theorem 7.3. (Efimov and Aleksandrov, 2020) Let there exist a locally Lipschitz continuous Lyapunov-Razumikhin function $V : \mathbb{R}^n \to \mathbb{R}_+$ with

(7.5) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and all $x \in \mathbb{R}^n$, and one of the following properties is satisfied:

(i) for some $\mu \in (0,1)$, $c > 0$, $\alpha' > 0$ and all $\varphi \in C_{[-\tau,0]}$:

$$\max_{\theta\in[-\tau,0]} V^{1-\mu}(\varphi(\theta)) \leq V^{1-\mu}(\varphi(0)) + c\tau \Rightarrow$$
$$D^+V(\varphi(0))\, f(\varphi) \leq -\alpha' V^{\mu}(\varphi(0)),$$

then the origin is globally FTS for the system (7.1), and for all $x_0 \in C_{[-\tau,0]}$ and $t \geq 0$:

$$|x(t,x_0)| \leq \max\{0, \alpha_1^{-1} \circ (\alpha_2^{1-\mu}(\|x_0\|)$$
$$- \min\{\alpha'(1-\mu), c\}t)^{\frac{1}{1-\mu}}\};$$

(ii) for some $\mu > 1$, $c > 0$, $\alpha' > 0$ and all $\varphi \in C_{[-\tau,0]}$:

$$\frac{1}{\left(\max_{\theta\in[-\tau,0]} V(\varphi(\theta))\right)^{1-\mu} + c\tau} \leq V^{\mu-1}(\varphi(0)) \Rightarrow$$
$$D^+V(\varphi(0))\, f(\varphi) \leq -\alpha' V^{\mu}(\varphi(0)),$$

then the system (7.1) is globally nearly FxTS at the origin, and for all $x_0 \in C_{[-\tau,0]}$ and $t \geq 0$:

$$|x(t,x_0)| \leq \alpha_1^{-1}\left(\frac{1}{\left(\alpha_2^{1-\mu}(\|x_0\|) + \min\{\alpha'(\mu-1), c\}t\right)^{\frac{1}{\mu-1}}}\right).$$

As we can conclude from the results of Proposition 7.3 and Theorem 7.3, the Lyapunov-Razumikhin approach can be used for estimation of the rate of convergence, but the conditions have to be formulated differently in accordance with the kind of decay.

Example 7.2. (Efimov and Aleksandrov, 2020) Consider a scalar example with $V(t) \in \mathbb{R}_+$ for all $t \geq 0$, and $V_0 \in C_{[-\tau,0]}$ with $\|V_0\| \leq 1$:

$$\dot{V}(t) \leq -aV^{\mu}(t) + bV^{\eta}(t)V^{\rho}(t-\tau),$$

where $a > 0$, $b > 0$, $\mu \in (0,1)$, $\rho > 0$ and $\eta \geq \mu$ are parameters. For

$c > 0$ the Lyapunov-Razumikhin relation is leads to:

$$V^{1-\mu}(t-\tau) < V^{1-\mu}(t) + c\tau \Rightarrow$$
$$\dot{V}(t) \leq -aV^{\mu}(t) + bV^{\eta}(t)\left(V^{1-\mu}(t) + c\tau\right)^{\frac{\rho}{1-\mu}}.$$

Applying Jensen's inequality,

$$\left(V^{1-\mu}(t) + c\tau\right)^{\frac{\rho}{1-\mu}} \leq \left(V^{\rho}(t) + (c\tau)^{\frac{\rho}{1-\mu}}\right) \times \begin{cases} 1 & \rho \in (0, 1-\mu] \\ 2^{\frac{\rho}{1-\mu}-1} & \rho > 1-\mu \end{cases},$$

we obtain

$$\begin{aligned} V^{1-\mu}(t-\tau) &< V^{1-\mu}(t) + c\tau \Rightarrow \\ \dot{V}(t) &\leq -aV^{\mu}(t) + \max\{1, 2^{\frac{\rho}{1-\mu}-1}\}b \\ &\quad \times [V^{\eta+\rho}(t) + V^{\eta}(t)(c\tau)^{\frac{\rho}{1-\mu}}] \\ &\leq -[a - \max\{1, 2^{\frac{\rho}{1-\mu}-1}\}b(1 + (c\tau)^{\frac{\rho}{1-\mu}})]V^{\mu}(t), \end{aligned}$$

and the conditions for FTS of Theorem 7.3 are verified locally for

$$a > \max\{1, 2^{\frac{\rho}{1-\mu}-1}\}b(1 + (c\tau)^{\frac{\rho}{1-\mu}}).$$

Let $V_0 \in C_{[-\tau,0]}$ in

$$\dot{V}(t) \leq -aV^{\mu}(t) + \frac{bV(t)V^{\mu-1}(t-\tau)}{1 + c\tau V^{\mu-1}(t-\tau)},$$

where $a > 0$, $b > 0$, $\mu > 1$, $c > 0$ are parameters. The Lyapunov-Razumikhin relation implies:

$$\frac{1}{V^{1-\mu}(t-\tau) + c\tau} < V^{\mu-1}(t) \Rightarrow$$
$$\dot{V}(t) \leq -(a-b)V^{\mu}(t),$$

and the conditions of Theorem 7.3 are verified for

$$a > b.$$

Example 7.3. Note that an FTS system (7.4) does not satisfy the conditions of Theorem 7.3, which is not a contradiction since it admits a discontinuous settling-time functional, while the estimate given in Theorem 7.3 implies its continuity.

7.1.5 Discussion

As we can conclude from the results of this section, FTS is not a natural type of behavior of time-delay systems, since the influence of past values of the state may block a finite-time settling of the trajectories at the origin, then additional structural conditions are needed. Control design approaches for finite-time stabilization of time-delay systems can be found in Moulay *et al.* (2008), Polyakov *et al.* (2015b), and Nekhoroshikh *et al.* (2020) and the simplest ways of obtaining accelerated regulation in this class of systems is by using prediction techniques to compensate the delays as in Karafyllis (2006), the theory of homogeneity (Zimenko *et al.*, 2017; Zimenko *et al.*, 2019) or the domination approach (Wang *et al.*, 2020b).

7.2 Partial differential equations and evolution models

The proofs of all claims given in this section can be found in Polyakov (2020).

7.2.1 Stability of evolution systems

The notions of finite-time and fixed-time stability of evolution systems are similar to the case of ODEs. *Below we denote $x(t, x_0)$ a solution of the system* (3.25), *i.e.*, $\dot{x} = Ax + f(x)$, where x_0 indicates the initial condition of the system and means

$$x(0, x_0) = x_0.$$

Definition 7.2 (Lyapunov stability). The origin of a dynamical system is said to be locally (globally) *Lyapunov stable* if there exists $\varepsilon \in \mathcal{K}$ (resp. $\varepsilon \in \mathcal{K}_\infty$) such that

$$\|x(t, x_0)\| \leq \varepsilon(\|x_0\|), \qquad t > t_0$$

for any solution $x(t, x_0)$ of (3.25) and any $x_0 \in U$, where $U \subset \mathbb{H}$ is a neighborhood of the origin (resp. $U = \mathbb{H}$).

If the origin of the system is *unstable* if it does not satisfy Definition 7.2. The following result is well-known for ODE (Bhat and Bernstein, 2000) and holds for evolution systems as well.

Proposition 7.4. If the origin of a dynamical system is Lyapunov stable then $x(t, \mathbf{0}) \equiv \mathbf{0}$ is the unique solution of the system (3.25) with the initial condition $x(0) = \mathbf{0}$.

Definition 7.3 (Finite-time stability). The origin of a dynamical system is said to be locally (globally) finite-time stable if it is locally (globally) Lyapunov stable and *finite-time attractive* in an open neighborhood U of the origin (resp. $U = \mathbb{H}$), *i.e.*, $\exists T : \mathbb{H} \to \mathbb{R}_+$ such that

$$\forall x_0 \in U \quad \Rightarrow \quad x(t, x_0) = \mathbf{0}, \;\; \forall t \geq T(x_0)$$

for any solution $x(t, x_0)$ of the system. The set U is called a domain of finite-time attraction.

Obviously, if T satisfies the latter definition then the function $T + T^+$ also does for any nonnegative T^+. Therefore, it is reasonable to consider a minimal functional T.

Definition 7.4. A functional $T : \mathbb{H} \to \mathbb{R}_+$ is called the *settling-time function* of the finite-time stable system, if T satisfies Definition 7.3 and the functional $T - \tilde{T}$ does not satisfy Definition 7.3 (with the same U) for any $\tilde{T} : \mathbb{H} \to \mathbb{R}_+ : \tilde{T} \neq \mathbf{0}$.

Finite-time stability always implies asymptotic stability. The settling-time function T of time-invariant finite-time stable system is independent of t_0. However, in contrast to asymptotic and Lyapunov stability, finite-time stability of time-invariant system, in general, does not imply uniform finite-time stability, which requires at least boundedness of the settling-time function in a neighborhood of the origin (see Bhat and Bernstein, 2000, p. 756).

Definition 7.5 (Uniform finite-time stability). The origin of a dynamical system is said to be locally (globally) uniformly finite-time stable if it is finite-time stable with $U \subset \mathbb{H}$ (resp. $U = \mathbb{H}$) and the settling time function $T : U \to \mathbb{R}$ is *locally bounded*, *i.e.*,

$$\forall y \in U, \quad \exists \varepsilon > 0 \quad \text{such that} \quad \sup_{\|x_0 - y\| < \varepsilon} T(x_0) < +\infty.$$

Definition 7.6 (Fixed-time stability)**.** The origin of a dynamical system is said to be locally (globally) fixed-time stable if it is locally (globally) uniformly finite-time stable with $U \subset \mathbb{H}$ (resp. $U = \mathbb{H}$) and the settling time function T is *bounded on* U, *i.e.*,

$$\exists T_{\max} > 0 : x(t, x_0) = 0, \quad t > t_0 + T_{\max}, \quad \forall t_0 \in \mathbb{R}, \quad \forall x_0 \in U.$$

It is worth stressing that in the infinite-dimensional case, the fixed-time stability can be discovered even for *linear* evolution systems.

Example 7.4. Let us consider the wave equation

$$u_{tt} = u_{xx}, \quad t > 0, \quad x \in [0, 1], \quad u : \mathbb{R}_+ \times [0, 1] \to \mathbb{R},$$

with the so-called "transparent" boundary condition (see, *e.g.*, Perrollaz and Rosier, 2014)

$$u_x(t, 0) = u_t(t, 0), \qquad u_x(t, 1) = -u_t(t, 1)$$

and the initial conditions

$$u(0, x) = \phi(x), \quad u_t(0, x) = \psi$$

from

$$\{(\phi, \psi) \in H^1((0, 1), \mathbb{R}) \times L^2((0, 1), \mathbb{R}) : \phi(0) + \phi(1) + \int_0^1 \psi(s) ds = 0\},$$

where L^2 and H^1 are Lebesgue and Sobolev spaces, respectively. The boundary conditions are transparent in the sense that any wave $u(t, x) = f_1(x - t)$ traveling to the right leaves the domain at $x = 1$ and does not generate any reflected wave. Any wave $u(t, x) = f_2(x + t)$ traveling to the left leave the domain at $x = 0$ similarly. Since any solution of the wave system is given by $u(t, x) = f_1(x - t) + f_2(x + t)$ then it vanishes after the time instant $t = 1$ independently of the initial condition.

7.2.2 Lyapunov characterization of finite-time stability of evolution equations

Let

$$\bar{D}^{+}\phi(t) = \limsup_{h\to 0^{+}} \frac{\phi(t+h)-\phi(t)}{h}$$

denotes the right-hand upper Dini derivative of the function $\phi : \mathbb{R} \to \mathbb{R}$ at the point $t \in \mathbb{R}$. Using Bolzano–Weierstrass Theorem it can be shown that a finite or infinite Dini derivative exists for any function $\phi : \mathbb{R} \to \mathbb{R}$ and any point $t \in \mathbb{R}$.

Recall that a function $\varphi : (a,b) \to \mathbb{R}$ is *decreasing* on (a,b), $a < b$ if

$$\forall t_1, t_2 \in (a,b) : t_1 \le t_2 \;\Rightarrow\; \varphi(t_1) \ge \varphi(t_2).$$

From the above definitions we conclude

$$\phi \text{ is decreasing on } (a,b) \qquad \Leftrightarrow \qquad \bar{D}^{+}\phi(t) \le 0, \quad \forall t \in (a,b)$$

Lyapunov function candidates in $\mathbb{R}^n$ are positive definite and proper (see, *e.g.*, Clarke *et al.*, 1998). Recall that a mapping $f : \mathbb{R}^n \to \mathbb{R}$ is proper if an inverse image of any compact set is a compact set. In the general case, closedness and boundedness is not sufficient for compactness in Hilbert spaces, and the properness in the classical sense may be too strong condition for Lyapunov function candidate. For "generalized" proper functions introduced in Definition 7.7 an inverse image of any compact set belongs to a closed bounded set (which may be not compact in the general case). Below the word "generalized" is omitted for shortness.

Definition 7.7. For a positive definite functional $V : \Omega \subset \mathbb{H} \to [0, +\infty)$:

- V is said to be proper at $\mathbf{0}$ (**locally proper**) if there exists $\underline{V}, \overline{V} \in \mathcal{K}$ such that

  $$\underline{V}(\|x\|) \le V(x) \le \overline{V}(\|x\|) \quad \text{for} \quad x \in \Omega \backslash \{\mathbf{0}\},$$

  where Ω is a neighborhood of $\mathbf{0}$.

- V is said to be **globally proper** if $\underline{V}, \overline{V} \in \mathcal{K}^{\infty}$.

Theorem 7.4. The origin of the system (3.25) is locally (globally) uniformly finite-time stable with a continuous at the origin settling-time function if and only if there exists a locally (globally) proper positive definite functional $V : \Omega \subset \mathbb{H} \to [0, +\infty)$ such that the inequality

$$\bar{D}^{+}V(x(t, x_0)) \leq -1, \qquad \forall t > 0 \tag{7.6}$$

holds for any solution $x(t, x_0)$ of (3.25) as long as

$$x(t, x_0) \in \Omega \backslash \{\mathbf{0}\},$$

where Ω is a neighborhood of the origin (resp. $\Omega = \mathbb{H}$). Moreover, the settling-time of the finite-time stable system (3.25) admits the estimate $T(x_0) \leq V(x_0)$ for all $x_0 \in U$, where $U \subset \Omega$ is a neighborhood of the origin (resp. $U = \Omega = \mathbb{H}$).

To prove the necessity, the following Lyapunov function $V = V_0 + T$ can be utilized, where T is the settling-time function and $V_0(x_0) = \sup_{\forall t>0} \|x(t, x_0)\|$. If a solution with $x(0, x_0) = x_0$ is not unique then the latter supremum has to be taken over all solutions with such initial condition. The proof of sufficiency is rather straightforward.

Under additional restrictions to V, the stability analysis can be done using only the operators A and f in the right-hand side of (3.25).

Corollary 7.5. Let the system (3.25) have a classical solution for any $x_0 \in \mathcal{D}(A)$ and all mild solutions of the system depend continuously of initial conditions on compact intervals of time. If there exists a locally (globally) proper locally Lipschitz continuous positive definite functional $V : \mathbb{H} \backslash \{\mathbf{0}\} \to \mathbb{R}$ such that V is Frechét differentiable on $\mathcal{D}(A) \cap \Omega \backslash \{\mathbf{0}\}$ and

$$DV(x)(Ax + f(x)) \leq -1, \quad \forall x \in \Omega \cap \mathcal{D}(A) \backslash \{\mathbf{0}\}, \tag{7.7}$$

then the origin of the system (3.25) is locally (globally) uniformly finite-time stable with some finite-time attraction domain $U \subset \Omega$, and the settling time $T(x_0)$ admits the estimate

$$T(x_0) \leq V(x_0), \quad \forall x_0 \in U,$$

where Ω and $U \subset \Omega$ are some neighborhoods of the origin (resp. $U = \Omega = \mathbb{H}$) and $DV(x) \in \mathcal{L}(\mathbb{H}, \mathbb{H})$ denotes the Frechét derivative of V at the point $x \in \mathcal{D}(A)$.

This result follows from Theorem 7.4 and the identity

$$\frac{d}{dt}V(x(t,x_0)) = DV(x)(Ax + f(x))|_{x=x(t,x_0)},$$

which holds for any classical solution of (3.25).

Example 7.5. Let us consider the evolution equation

$$\dot{x} = Ax + f(x), \quad t > 0, \quad x(0) = x_0 \in \mathbb{H}, \tag{7.8}$$

where

$$f(x) = -\frac{x}{\|x\|^{\alpha}}, \quad x \in \mathbb{H}, \quad \alpha \in (0,1), \quad \|x\| = \sqrt{\langle x, x\rangle},$$

the operator $A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}$ generates a strongly continuous semigroup of linear bounded operators on a real Hilbert space $\mathbb{H}$. Assume also that A is a dissipative operator, *i.e.*,

$$\langle Ax, x\rangle \leq 0, \quad x \in \mathcal{D}(A).$$

The functional $V : \mathbb{H} \to [0, +\infty)$ given by

$$V(x) = \frac{1}{\alpha}\|x\|^{\alpha}$$

is Frechét differentiable on $\mathbb{H}\backslash\{\mathbf{0}\}$ and for any $x \in D(A)\backslash\{\mathbf{0}\}$ we have

$$\dot{V}(x) = \sup_{y\in F(x)} DV(x)(Ax + y) = DV(x)\left(Ax - \frac{x}{\|x\|^{\alpha}}\right)$$

$$= \frac{1}{\|x\|^{2-\alpha}}\left\langle Ax - \frac{x}{\|x\|^{\alpha}}, x\right\rangle = \frac{\langle Ax, x\rangle}{\|x\|^{2-\alpha}} - 1 \leq -1.$$

The origin of the considered evolution system is globally finite-time stable and the settling-time T admits the estimate $T(x_0) \leq \|x_0\|$ for any $x_0 \in \mathbb{H}$.

In the infinite-dimensional case, a setting-time function T may be continuous, vanishing, but not proper. This means, in the general case, we cannot assume $V = T$ in Theorem 7.4.

Example 7.6 (Polyakov, 2019)**.** Let $\mathbb{H} = L^2((-1,1),\mathbb{R})$ and the operator A in the system (7.8) be defined as follows

$$Az = \frac{\partial^2}{\partial z^2}, \quad z \in \mathcal{D}(A)$$

with the domain

$$D(A) = H^2((-1,1),\mathbb{R}) \cap H_0^1((-1,1),\mathbb{R}).$$

It is easy to see that $\phi_i = \sin(\pi i z) \in D(A), z \in [-1,1]$ is an eigenvector of the operator A:

$$A\phi_i = -\lambda_i x_i, \quad \lambda_i = \pi^2 i^2, \quad i = 1, 2...$$

Being an orthonormal basis in the separable Hilbert space $\mathbb{H}$, the eigenvectors ϕ_i allow us to represent any $y \in \mathcal{D}(A)$ in the form

$$y = \sum_{i=1}^{\infty} x_i \phi(x). \tag{7.9}$$

Taking into account $\langle \phi_i, \phi_j \rangle = 0$ for $i \neq j$ and $\langle \phi_i, \phi_i \rangle = 1$ we derive

$$\langle \phi_i, Ay \rangle = -\lambda_i x_i, \quad \langle \phi_i, f(y) \rangle = -\frac{x_i}{\sqrt{\langle y,y \rangle}} = -\frac{x_i}{\sqrt{\sum_{i=1}^{+\infty} x_i^2}}.$$

Therefore, any classical solution of (7.8) admits the representation (7.9), where the time-varying functions x_i satisfy the following (infinite) system of ODEs

$$\dot{x}_i = -\left(\lambda_i + \frac{1}{\sqrt{\sum_{i=1}^{+\infty} x_i^2}}\right) x_i, \quad i = 1, 2, \ldots.$$

Since the operator A is dissipative then using Example 7.5 it can be shown that the considered system is finite-time stable, the settling-time T is continuous at the origin and $T(\mathbf{0}) = 0$.

Let us consider the sequence of initial conditions $x(0) = y_0^j = \phi_j \in \mathcal{D}(A)$, $j = 1, 2, \ldots$. Obviously, $\|y_0^j\| = 1$ for all $j \geq 1$, the corresponding solution y^j has the form $y_j(t) = x_j(t)\phi_j$ and $\|y_j(t)\| = x_j(t)$, where

$$\dot{x}_j = -\left(\lambda_j + \tfrac{1}{|x_j|}\right) x_j, \quad x_j(0) = 1.$$

Simple computations show

$$T(y_j) = \frac{\ln(1+\lambda_j)}{\lambda_j} \to 0 \quad \text{as} \quad j \to +\infty.$$

while $\|y_0^j\| = 1$ for all $j \geq 1$. The latter means that $\sup_{\|x_0\|=1} T(x_0) = 0$, *i.e.*, T is not a proper function and it cannot be utilized as a Lyapunov function in Theorem 7.4.

Corollary 7.6. If under conditions of Corollary 7.5 there exists a locally (globally) proper locally Lipschitz continuous positive definite functional $V : \mathbb{H}\backslash\{\mathbf{0}\} \to \mathbb{R}$ such that V is Frechét differentiable on $\mathcal{D}(A) \cap \Omega\backslash\{\mathbf{0}\}$ and

$$DV(x)(Ax + f(t,x)) \leq -q(1 + V^2(x)), \quad \forall x \in D(A) \cap \Omega\backslash\{\mathbf{0}\},\ \forall t \in \mathbb{R}, \tag{7.10}$$

then the origin of the system (3.25) is locally (globally) uniformly fixed-time stable with an attraction domain $U \subset \Omega$ (resp. $U = \mathbb{H}$) and the settling time $T(x_0)$ admits the estimate

$$T(x_0) \leq \frac{\pi}{2q}$$

for all $x_0 \in \mathbb{H}$.

The converse Lyapunov theorem of fixed-time stability is still an open problem in the general case. A particular case (see, Theorem 2.5) is studied in Lopez-Ramirez *et al.*, 2019 for $\mathbb{H} = \mathbb{R}^n$, where it is assumed that the settling-time function is, at least, locally proper. This is not the case for an unbounded operator A in (3.25) (see Example 7.6).

Example 7.7. Let us consider the evolution equation

$$\dot{x} = Ax - \tfrac{x}{\|x\|} - \|x\|x, \quad x \in \mathbb{H}$$

in a Hilbert space $\mathbb{H}$, where A is assumed to be a dissipative operator, *i.e.*,

$$\langle Ax, x\rangle \leq 0, \quad \forall x \in \mathcal{D}(A).$$

Obviously, selecting

$$V(x) = \|x\| = \sqrt{\langle x, x\rangle}, \quad x \in \mathbb{H}$$

we derive

$$\dot{V}(x) = \tfrac{\langle \dot{x}, x\rangle}{\|x\|} = \langle Ax, x\rangle - 1 - V^2, \quad \forall x \in \mathcal{D}(A)\backslash\{\mathbf{0}\}.$$

The latter means that the considered system is globally uniformly fixed-time stable and the settling-time admits the estimate

$$T(x_0) \leq \tfrac{\pi}{2}.$$

7.2.3 Finite-time stabilization of a linear evolution equation in a Hilbert space

Let us consider the following control system

$$\dot{x}(t) = Ax(t) + Bu(x(t)), \quad t > 0 \tag{7.11}$$

$$x(0) = x_0, \tag{7.12}$$

where $A : \mathcal{D}(A) \subset \mathbb{B} \to \mathbb{B}$ is a (possibly unbounded) closed linear operator with the domain $\mathcal{D}(A)$ dense in $\mathbb{B}$, $B : \mathbb{X} \to \mathbb{B}$ is a linear bounded operator, $\mathbb{B}$ is a real Banach space and $\mathbb{X}$ is a normed vector space, $x(t)$ is the system state, $u : \mathbb{B} \to \mathbb{X}$ is a (*locally or globally*) bounded feedback, *i.e.*, interior or distributed control.

Theorem 7.7 (Polyakov *et al.*, 2018, Polyakov, 2020)**.** Let $A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}$ be a generator of a strongly continuous semigroup Φ of linear bounded operators on $\mathbb{H}$, $B : \mathbb{X} \to \mathbb{H}$ be a linear bounded operator and

A) $\mathbf{d}_1$ be a uniformly continuous group of linear bounded operators in $\mathbb{H}$ with the generator $G_{\mathbf{d}_1} \in \mathcal{L}(\mathbb{H}, \mathbb{H})$;

B) $\mathbf{d}$ be a strongly continuous strictly monotone linear dilation in $\mathbb{H}$ such that $\mathcal{D}(A) \subset \mathcal{D}(G_{\mathbf{d}})$, where $G_{\mathbf{d}} : \mathcal{D}(G_{\mathbf{d}}) \subset \mathbb{H} \to \mathbb{H}$ is the generator of the dilation group $\mathbf{d}$;

C) there exist a linear bounded operator $K : \mathbb{H} \to \mathbb{X}$ such that

$$\exists \rho > 0 : \langle (A + BK + \rho G_{\mathbf{d}})x, x\rangle \leq 0, \quad \forall x \in \tilde{\mathcal{D}}, \tag{7.13}$$

where a set $\tilde{\mathcal{D}} \subset \mathcal{D}(A)$ is dense in $\mathcal{D}(A)$.

D) the operator A be $\mathbf{d}$-homogeneous of a degree $\mu \in \mathbb{R}$ with $\mu > -\beta$ and

$$\mathbf{d}(s)BK\mathbf{d}_1(s) = BK\mathbf{d}(s), \quad \forall s \in \mathbb{R}.$$

Then the following holds.

- The feedback control $u : \mathbb{H} \to \mathbb{X}$ defined as

$$u(x) = \begin{cases} \|x\|_{\mathbf{d}}^{\mu} K\mathbf{d}_1(-\ln(\|x\|_{\mathbf{d}}))x & \text{if} \quad x \neq \mathbf{0}, \\ \mathbf{0} & \text{if} \quad x = \mathbf{0} \end{cases} \tag{7.14}$$

  is locally Lipschitz continuous on $\mathbb{H}\backslash\{\mathbf{0}\}$ and Fréchet differentiable on $\mathcal{D}(G_{\mathbf{d}})\backslash\{\mathbf{0}\}$. Moreover,

  - if $\beta + \mu > \gamma_{\max}$ then u is continuous at $\mathbf{0} \in \mathbb{H}$;
  - if $\beta + \mu \geq \gamma_{\max}$ then

$$\sup_{\|x\|\leq 1} \|u(x)\| \leq \|K\|;$$

  - if $\exists M \geq 1 : \|\mathbf{d}(s)\| \leq M e^{(-\mu+\gamma_{\min})s}$, $\forall s > 0$ then

$$\sup_{\|x\|\geq 1} \|u(x)\| \leq M\|K\|,$$

  where $\beta := \inf_{z\in S\cap\mathcal{D}(G_{\mathbf{d}})} \langle G_{\mathbf{d}}z, z\rangle > 0$, $\gamma_{\min} := \inf_{z\in S} \langle G_{\mathbf{d}_1}z, z\rangle$ and $\gamma_{\max} := \sup_{z\in S} \langle G_{\mathbf{d}_1}z, z\rangle$.

- The closed-loop system (7.11)- (7.14) is $\mathbf{d}$-homogeneous of degree μ.

- The closed-loop system (7.11)- (7.14) has

  - a unique mild solution defined on $[0, +\infty)$ for any $x_0 \in \mathbb{H}$;
  - a unique locally Lipschitz continuous strong solution defined on $[0, +\infty)$ for any $x_0 \in \mathcal{D}(A)$.

  - a unique classical solution defined on $[0, +\infty)$ for any $x_0 \in \mathcal{D}(A)$ provided that $BK\mathcal{D}(A) \subset \mathcal{D}(A)$.

- The origin of the closed-loop system (7.11)- (7.14) is
  - globally finite-time stable for $\mu < 0$ and the settling-time functional $T : \mathbb{H} \to [0, +\infty)$ admits the estimate

$$T(x_0) \leq \frac{\|x_0\|_{\mathbf{d}}^{-\mu}}{-\rho\mu}; \tag{7.15}$$

  - globally exponentially stable for $\mu = 0$ and

$$\|x(t)\|_{\mathbf{d}} \leq \|x_0\|_{\mathbf{d}} e^{-\rho t}; \tag{7.16}$$

  - globally nearly fixed-time stable for $\mu > 0$ and

$$\forall r > 0 \quad : \quad \|x(t)\|_{\mathbf{d}} \leq r, \quad \forall t \geq \frac{1}{\rho\mu r^{\mu}}. \tag{7.17}$$

  independently of initial conditions.

- The canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is a Lyapunov function of the closed-loop system (7.11)- (7.14) such that

$$\frac{d}{dt}\|x(t)\|_{\mathbf{d}} \overset{a.e.}{\leq} -\rho\|x(t)\|_{\mathbf{d}}^{1+\mu}, t > 0 \tag{7.18}$$

  for any strong solution of the closed-loop system (7.11)- (7.14).

The condition $\mathcal{D}(A) \subset \mathcal{D}(G_{\mathbf{d}})$ in the latter theorem is required for differentiability of the canonical homogeneous on $\mathcal{D}(A)$. If the canonical homogeneous norm in differentiable on whole $\mathbb{H}\backslash\{\mathbf{0}\}$, then the mentioned condition can be omitted.

Corollary 7.8 (Polyakov, 2020, p. 279)**.** If a linear dilation $\mathbf{d}$ in $\mathbb{H} = L^2(\mathbb{R}^n, \mathbb{R}^m)$ is given by (3.17) then the condition $\mathcal{D}(A) \subset \mathcal{D}(G_{\mathbf{d}})$ can be omitted in Theorem 7.7.

7.2.4 Examples of finite-time and fixed-time stabilization of PDEs

Homogeneous stabilization of heat equation on $\mathbb{R}^n$

Let $\mathbb{H} = \mathbb{X} = L^2(\mathbb{R}^n, \mathbb{R})$ and $A = \Delta : \mathcal{D}(\Delta) \subset \mathbb{H} \to \mathbb{H}$ be the Laplace operator and $B = I$ be an identity operator, where $\mathcal{D}(\Delta) = H^2(\mathbb{R}^n, \mathbb{R}) \cap H_0^1(\mathbb{R}^n, \mathbb{R})$ is the domain of Δ

Let $\mathbf{d}$ be selected in the form (3.17) with $G_\alpha = \alpha > -\frac{n}{4}$ and $G_\beta = \beta I_n$, where $\beta = -\frac{1}{2}$. In this case, from (3.20) we derive have $\|x\|_{\mathbf{d}} = \|x\|^{\frac{4}{4\alpha+n}}$.

In Example 3.2 we have shown that the Laplace operator is $\mathbf{d}$-homogeneous of the degree $2\beta = -1$. Taking into account Corollary 7.8 we conclude that Condition B) of Theorem 7.7 holds.

Let us select in (7.14) $K \in \mathcal{L}(\mathbb{H}, \mathbb{X})$ as follows

$$Kx = -\left(\alpha + \tfrac{n}{4}\right) x, \quad x \in \mathbb{H}.$$

The generator of the dilation $\mathbf{d}$ is given by (see Lemma 3.18)

$$(G_{\mathbf{d}} x)(z) = \alpha z(x) - \frac{1}{2} z \cdot \nabla x(z)$$

where $z = (z_1, ..., z_n)^\top \in \mathbb{R}^n$ and $x \in \mathcal{D}(G_{\mathbf{d}})$.

In this case, for any $x \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ and any open bounded connected set $\Omega \subset \mathbb{R}^n$ with a smooth boundary, which contains a support of x, we have

$$\langle x, (A + BK + G_{\mathbf{d}})x \rangle =$$

$$\langle x, \Delta x \rangle + \langle x, BKx \rangle + \alpha \langle x, x \rangle - \tfrac{1}{2} \int_\Omega \sum_{i=1}^n z_i x(z) \tfrac{\partial x(z)}{\partial z_i} dz =$$

$$\langle x, \Delta x \rangle - \tfrac{n}{4} \langle x, x \rangle - \tfrac{1}{4} \int_\Omega \sum_{i=1}^n z_i \tfrac{\partial x^2(z)}{\partial z_i} dz =$$

$$-\int_\Omega |\nabla x(z)|^2 dz + \frac{n}{4} \langle x, x \rangle - \frac{n}{4} \int_\Omega x^2(z) dz \le 0,$$

where the integration by parts (for the first and the third term) has been utilized on the last step. Since $C_c^\infty(\mathbb{R}^n, \mathbb{R})$ is dense in L^2 then Condition C) of Theorem 7.7 is fulfilled.

Finally, selecting the group $\mathbf{d}_1(s) = I$ (with the generator $G_{\mathbf{d}_1} = \mathbf{0}$) we conclude that all conditions of Therem 7.7 are satisfied. This means that the control

$$u(x) = -\frac{4\alpha + n}{4} \frac{x}{\|x\|^{\frac{4}{4\alpha+n}}}$$

is continuous for $\alpha > 1 - \frac{n}{4}$, globally bounded for $\alpha = 1 - \frac{n}{4}$ and stabilizes the heat system

$$\dot{x} = \Delta x + u(x)$$

in a finite time $T(x_0) \leq \frac{4\|x_0\|^{\frac{4}{4\alpha+n}}}{4\alpha+n}$. For $\alpha = 1 - \frac{n}{4}$ the control u is discontinuous at the origin, so mild Filippov solutions have to be considered for $t \geq T(x_0)$ in this case.

Homogeneous control for a wave equation on a line

Let

$$\mathbb{H} = H^1(\mathbb{R}, \mathbb{R}) \times L^2(\mathbb{R}, \mathbb{R})$$

with an inner product to be defined below.

Let us consider a wave equation represented in the form (7.11) using the notations:

$$A = \begin{pmatrix} O & I \\ \frac{\partial^2}{\partial z^2} & O \end{pmatrix} : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H},$$

where

$$\mathcal{D}(A) = H^2(\mathbb{R}, \mathbb{R}) \times H^1(\mathbb{R}, \mathbb{R}),$$

$$O \in \mathcal{L}(L^2(\mathbb{R}, \mathbb{R}), L^2(\mathbb{R}, \mathbb{R})) \quad \text{and} \quad I \in \mathcal{L}(L^2(\mathbb{R}, \mathbb{R}), L^2(\mathbb{R}, \mathbb{R}))$$

are zero and identity operators, respectively, and

$$B = \begin{pmatrix} O \\ I \end{pmatrix} : L^2(\mathbb{R}, \mathbb{R}) \to \mathbb{H}.$$

Let the inner product in $\mathbb{H}$ be defined as follows

$$\langle x, \tilde{x} \rangle = \int_{\mathbb{R}} x^\top(z) P \tilde{x}(z) + p_{22} \frac{\partial x_1}{\partial z} \frac{\partial \tilde{x}_1}{\partial z} dz,$$

where

$$P := \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = Q^{-1}$$

with

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

being a positive definite solution of the system of linear matrix inequalities (LMIs) and equations:

$$\left(\begin{smallmatrix} \nu-1.5\mu & 1 \\ 0 & \nu-0.5\mu \end{smallmatrix}\right)Q + Q\left(\begin{smallmatrix} \nu-1.5\mu & 0 \\ 1 & \nu-0.5\mu \end{smallmatrix}\right) + y^\top b^\top + by = 0, \tag{7.19}$$

$$\left(\begin{smallmatrix} \nu-1.5\mu & 0 \\ 0 & \nu-0.5\mu \end{smallmatrix}\right)Q + Q\left(\begin{smallmatrix} \nu-1.5\mu & 0 \\ 0 & \nu-0.5\mu \end{smallmatrix}\right) \succ 0, \quad Q \succ 0, \tag{7.20}$$

where $\nu > \max\{0, 1.5\mu\}$, $b = (0\ \ 1)^\top \in \mathbb{R}^2$, $y = (y_1\ \ y_2) \in \mathbb{R}^{1\times 2}$.

The operator A is a generator of a strongly continuous semigroup of linear bounded operators on $\mathbb{H}$ (see, *e.g.*, Pazy, 1983, Section 7.4 for more details).

One can be shown that this system of LMIs is always feasible with respect to Q and y. Since from (7.19) we conclude

$$(\nu - 1.5\mu)q_{11} + q_{12} = 0$$

then taking into account $P = Q^{-1} \succ 0$ it is easy to see that

$$0 < (\nu - 1.5\mu)p_{22} = p_{12}.$$

Let the operator $K : \mathbb{H} \to L^1(\mathbb{R}, \mathbb{R})$ be defined as

$$Kx := yQ^{-1}x, \quad x \in \mathbb{H},$$

where the pair (Q, y) is a solution of the LMIs (7.19), (7.20).

A) Let the uniformly continuous semigroup $\mathbf{d}_1$ of linear bounded operators on $\mathbb{H}$ be defined as follows

$$\mathbf{d}_1(s)x = \begin{pmatrix} e^{-\mu s} & 0 \\ 0 & 1 \end{pmatrix} x.$$

Its generator

$$G_{\mathbf{d}_1}x = \begin{pmatrix} -\mu & 0 \\ 0 & 0 \end{pmatrix} x,$$

obviously, satisfies the condition A) of Theorem 7.7 with $\gamma_{\min} = \min\{0, -\mu\}$ and $\gamma_{\max} = \max\{0, -\mu\} \geq 0$.

B) Let us introduce the dilation $\mathbf{d}$ on $\mathbb{H}$ as follows

$$(\mathbf{d}(s)x)(z) = \left(\begin{smallmatrix} e^{s(\nu-\mu)} & 0 \\ 0 & e^{s\nu} \end{smallmatrix}\right) x(e^{\mu s} z), \quad x \in \mathbb{H}, \quad z \in \mathbb{R}.$$

The dilation $\mathbf{d}$ has the generator $G_{\mathbf{d}} : \mathcal{D}(G_{\mathbf{d}}) \subset \mathbb{H} \to \mathbb{H}$ defined as

$$G_{\mathbf{d}} x = \left(\begin{smallmatrix} \nu-\mu & 0 \\ 0 & \nu \end{smallmatrix}\right) x + \mu z \frac{\partial x}{\partial z},$$

where $\mathcal{D}(G_{\mathbf{d}}) = \left\{ x \in \mathbb{H} : z\frac{\partial x}{\partial z} \in \mathbb{H} \right\}$.

According to Proposition 3.4 the dilation $\mathbf{d}$ is strictly monotone on $\mathbb{H}$ if there exists $\beta > 0$ such that $\langle G_{\mathbf{d}} x, x\rangle \geq \beta \|x\|^2$ for any $x \in C_c^\infty(\mathbb{R},\mathbb{R}) \times C_c^\infty(\mathbb{R},\mathbb{R})$, where $\|x\|^2 = \sqrt{\langle x, x\rangle}$. Using integration by parts we derive

$$\langle G_{\mathbf{d}} x, x\rangle = \int_{\mathbb{R}} x^\top(z) P \left(\left(\begin{smallmatrix} \nu-\mu & 0 \\ 0 & \nu \end{smallmatrix}\right) x(z) + \mu z \tfrac{\partial x(s)}{\partial z} \right) dz +$$

$$p_{22} \int_{\mathbb{R}} (\nu-\mu) \left(\tfrac{\partial x_1(z)}{\partial z}\right)^2 + \mu \tfrac{\partial x_1(z)}{\partial z} \tfrac{\partial}{\partial z} \left(z \tfrac{\partial x_1(z)}{\partial z} \right) dz =$$

$$\int_{\mathbb{R}} x^\top(z) P \left(\begin{smallmatrix} \nu-\mu & 0 \\ 0 & \nu \end{smallmatrix}\right) x(z) dz + \tfrac{\mu}{2} \int_{\mathbb{R}} z \frac{\partial}{\partial z} \left(x^\top(z) P x(z) \right) dz +$$

$$p_{22} \nu \int_{\mathbb{R}} \left(\tfrac{\partial x_1(z)}{\partial z}\right)^2 dz + \tfrac{\mu}{2} \int_{\mathbb{R}} z \frac{\partial}{\partial z} \left(\tfrac{\partial x_1(z)}{\partial z}\right)^2 dz =$$

$$\int_{\mathbb{R}} x^\top(z) P \left(\begin{smallmatrix} \nu-1.5\mu & 0 \\ 0 & \nu-0.5\mu \end{smallmatrix}\right) x(z) + p_{22}(\nu - 0.5\mu) \left(\tfrac{\partial x_1(z)}{\partial z}\right)^2 dz \geq \beta \|x\|^2,$$

where

$$\beta \geq \min \left\{ \lambda_{\min} \left(P^{\frac{1}{2}} \left(\begin{smallmatrix} \nu-\frac{3\mu}{2} & 0 \\ 0 & \nu-\frac{\mu}{2} \end{smallmatrix}\right) P^{-\frac{1}{2}} + P^{-\frac{1}{2}} \left(\begin{smallmatrix} \nu-\frac{3\mu}{2} & 0 \\ 0 & \nu-\frac{\mu}{2} \end{smallmatrix}\right) P^{\frac{1}{2}} \right), \nu - \frac{\mu}{2} \right\}$$

The LMI (7.20) implies $\beta > 0$. The dilation $\mathbf{d}$ has the form (3.17) and Condition B) of Theorem 7.7 is fulfilled in the view of Corollary 7.8.

C) Given $x = (x_1, x_2)^\top \in C_c^\infty(\mathbb{R},\mathbb{R}) \times C_c^\infty(\mathbb{R},\mathbb{R})$ and $A_0 = \left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right)$ we have

$$\langle (A + BK + G_{\mathbf{d}})\, x, x\rangle = \langle (A_0 + BK + G_{\mathbf{d}})\, x, x\rangle + \left\langle \left(\begin{smallmatrix} \mathbf{0} \\ \frac{\partial^2 x_1}{\partial z^2} \end{smallmatrix} \right), x \right\rangle.$$

Using (7.19) we derive

$$\langle x, (A_0 + BK + G_{\mathbf{d}}) x\rangle = p_{22} \int_{\mathbb{R}} \tfrac{\partial x_1(z)}{\partial z} \tfrac{\partial x_2(z)}{\partial z} dz + p_{22}(\nu - 0.5\mu) \int_{\mathbb{R}} \left(\tfrac{\partial x_1(z)}{\partial z}\right)^2 dz.$$

Using integration by parts we derive

$$\left\langle x, \left(\begin{smallmatrix} \mathbf{0} \\ \frac{\partial^2 x_1}{\partial z^2} \end{smallmatrix} \right) \right\rangle = \int_{\mathbb{R}} -p_{12} \left(\frac{\partial x_1(z)}{\partial z}\right)^2 - p_{22} \frac{\partial x_1(z)}{\partial z} \frac{\partial x_2(z)}{\partial z} dz,$$

Therefore, taking into account the identity $0 < (\nu - 1.5\mu)p_{22} = p_{12}$ we conclude

$$\langle x, (A + BK + G_{\mathbf{d}})\, x \rangle = \mu p_{22} \int_{\mathbb{R}} \left(\frac{\partial x_1(z)}{\partial z}\right) dz.$$

For $\mu \leq 0$ we, obviously, have $\langle x, (A + BK + G_{\mathbf{d}})\, x \rangle \leq 0$. On the other hand, if $\nu \geq 2.5\mu > 0$ then using the representation for $\langle x, G_{\mathbf{d}} x \rangle$ we derive

$$\langle x, (A + BK + 0.5 G_{\mathbf{d}})\, x \rangle \leq -\tfrac{p_{22}(\nu - 2.5\mu)}{2} \int_{\mathbb{R}} \left(\frac{\partial x_1(z)}{\partial z}\right) dz \leq 0.$$

Therefore, the condition C) of Theorem 7.7 holds for any $\mu \in \mathbb{R}$ and any $\nu \geq \max\{0, 2.5\mu\}$.

D) Finally, $\mathbf{d}(s) BK \mathbf{d}_1(s) = BK \mathbf{d}(s)$ for any $s \in \mathbb{R}$ and the operator A is $\mathbf{d}$-homogeneous of the degree μ. Indeed,

$$(A\mathbf{d}(s)x)(z) = A \begin{pmatrix} e^{(\nu-\mu)s} x_1(e^{\mu s} z) \\ e^{\nu s} x_2(e^{\mu s} z) \end{pmatrix} = \begin{pmatrix} e^{\nu s} x_2(e^{\mu s} z) \\ e^{(\nu+\mu)s} \left.\frac{\partial^2 x_1(q)}{\partial q^2}\right|_{q = e^{\mu s} z} \end{pmatrix} = e^{\mu s} (\mathbf{d}(s) A x)(z)$$

and the condition D) of Theorem 7.7 is fulfilled.

Therefore, the homogeneous feedback control of the form (7.14) steers all trajectories of the wave system to

- the origin in a finite time if $\mu < 0$;
- a neighborhood of the origin in a fixed time independently of the initial state if $\mu > 0$.

In the general case, the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is defined implicitly, see (3.20). For $\mu = -1$, $\nu = \frac{1}{2}$ the canonical homogeneous norm can be found as a unique positive solution of the quartic equation

$$V^4 = aV^2 + bV + c$$

with $V = e^{s_x} = \|x\|_{\mathbf{d}}$,

$$a = p_{22} \int_{\mathbb{R}} x_2^2(z) + (x_1'(z))^2 dz, \quad b = 2p_{12} \int_{\mathbb{R}} x_1(z) x_2(z) dz, \quad c = p_{11} \int_{\mathbb{R}} x_1^2(z) dz.$$

In this case $\|x\|_{\mathbf{d}}$ can be computed using Ferrari formulas, *i.e.*, the homogeneous finite-time stabilizing feedback (7.14) for the wave equation admits an explicit representation.

Distributed finite-time control of the heat equation on $[0, 1]$

Inspired by Polyakov *et al.* (2018), let us consider a distributed finite-time control for the following heat system on $[0, 1]$:

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} + \phi(z)u(t, z),\ x(t, 0) = x(t, 1) = 0,\ x(0, z) = x_0(z),$$

where x is the system state, u is the distributed control, $\phi : [0, 1] \to \mathbb{R}_+$ is a continuous function such that

$$c\, z^2 \leq \phi(z) \quad \text{for} \quad z \in [0, 1], \tag{7.21}$$

for some $c > 0$. The function ϕ represents possibly non-uniform feedback gain on $[0, 1]$. The case $\phi \equiv 1$ is studied in Pisano *et al.* (2011).

Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ be, respectively, the inner product and the norm in $L^2((0, 1), \mathbb{R})$. Let us define the control u law as follows

$$u = -\frac{c^{-1} x}{\sqrt{\|x\|}}.$$

Using (7.21) and the integration by parts, for any classical solution of the system we derive

$$\frac{1}{2}\frac{d\|x\|^2}{dt} = \left\langle x, \frac{\partial^2 x}{\partial z^2} \right\rangle + \langle x, \phi(z)u(x) \rangle =$$

$$-\left\| \frac{\partial x}{\partial z} \right\|^2 - \frac{c^{-1}}{\sqrt{\|x\|}} \langle x, \phi(z)x \rangle \leq -\left\| \frac{\partial x}{\partial z} \right\|^2 - \frac{1}{\sqrt{\|x\|}} \|zx\|^2 =$$

$$-\left\| \frac{\partial x}{\partial z} \right\|^2 - 2\left\langle \frac{zx}{\sqrt[4]{\|x\|}}, \frac{\partial x}{\partial z} \right\rangle - \left\| \frac{zx}{\sqrt[4]{\|x\|}} \right\|^2 + 2\frac{1}{\sqrt[4]{\|x\|}} \left\langle zx, \frac{\partial x}{\partial z} \right\rangle =$$

$$-\left\| \frac{\partial x}{\partial z} + \frac{1}{\sqrt[4]{\|x\|}} zx \right\|^2 - \|x\|^{\frac{3}{4}}.$$

Hence, $\|x(t)\| = 0$ for $\forall t \geq \frac{4}{5}\|x_0\|^{\frac{5}{4}}$.

Boundary finite-time control of the heat equation on $[0, 1]$

The distributed control of a heat system on $[0, 1]$ can designed as a trivial conclusion of Example (7.7). Boundary finite-time stabilization of the heat system is much more complicated problem (Coron and Nguyen, 2017; Polyakov *et al.*, 2017; Espitia *et al.*, 2019).

Let us consider the heat system

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \quad v(0) = v_0 \tag{7.22}$$

with boundary control ζ

$$(v(t))(0) = 0 \qquad (v(t))(1) = \zeta(t), t > 0 \tag{7.23}$$

where $v(t) \in L^2((0,1),\mathbb{R})$.

The control aim is to steer the state $v(t)$ of the system (7.22), (7.23) to zero in a finite time by means of a linear switching feedback control

$$\zeta(t) = K_{\sigma(t)} v(t), \tag{7.24}$$

where $K_i : L^2((0,1),\mathbb{R}) \to \mathbb{R}$, $i \in \mathbb{Z}$ is a family of linear bounded functionals and $\sigma : \mathbb{R}_+ \to \mathbb{Z}$ is a *state dependent* switching function governed by the discrete equation

$$\sigma(t) = G(\sigma(t^-), v(t^-)), \tag{7.25}$$

with $G : \mathbb{Z} \times L^2((0,1),\mathbb{R}) \to \mathbb{Z}$, $t^- = t + 0^-$ and $\sigma(0) \in \mathbb{Z}$.

The hybrid (switched) linear system (7.22) - (7.25) has two components: $v(t)$ - continuous state and $\sigma(t)$ - discrete state. Therefore, its solution is a tuple $(v,\sigma) : v \in C^0([0,T), L^2((0,1),\mathbb{R}))$ and $\sigma : (0,T) \to \mathbb{Z}$ such that $\sigma(t)$ satisfies the discrete equation (7.25) for all $t \in (0,T)$ and v is a solution to the heat system (7.22) - (7.24) understood in the weak sense:

$$-\int_0^1 v_0(x)\xi(0,x)dx - \int_0^T \int_0^1 v(t,x)\xi_t(t,x)dxdt$$
$$+ \int_0^T K_{\sigma(t)} v(t,\cdot)\xi_x(t,1)dt - \int_0^T\int_0^1 v(t,x)\xi_{xx}(t,x)dxdt = 0$$

for all $\xi \in C^2([0,T) \times [0,1])$ with compact support in $[0,T) \times [0,1]$ such that ξ vanishes at $[0,T) \times \{0,1\}$.

We refer the reader to Coron and Nguyen (2017) for more details about existence of solutions and finite-time stabilization of (7.22) -

(7.24) in the case of time-dependent switchings. Information about switched systems can be found in Liberzon (2003).

Given positive number $\lambda > 0$, the backstepping approach (see, *e.g.*, Krstic and Smychlyaev, 2008, Coron and Nguyen, 2017 for the details) introduces the boundary control

$$\zeta(t) = \int_0^1 k(1, y, \lambda) v(t, y) dy \tag{7.26}$$

where

$$k(x, y, \lambda) = -\lambda y \frac{I_1(\sqrt{\lambda(x^2 - y^2)})}{\sqrt{\lambda(x^2 - y^2)})}$$

and I_m with $m \in \mathbb{Z}$ is the modified Bessel function of the first kind (see, *e.g.*, Watson, 1996). If we denote

$$(F(\lambda)u)(x) = -\int_0^x k(x, y, \lambda) u(y) dy \tag{7.27}$$

then the state transformation $u = v + F(\lambda)v$ applied to the system (7.22), (7.23) yields

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \lambda u, \qquad u(t, 0) = 0, \quad u(t, 1) = 0. \tag{7.28}$$

The *inverse transformation* is defined as

$$v = (I + F(-\lambda))u,$$

where $u, v \in L^2((0, 1), \mathbb{R})$. We refer the reader to Krstic and Smychlyaev (2008) and Coron and Nguyen (2017) for more details about backstepping transformation.

It is easy to see (using the Lyapunov function defined as $V(u) = \|u\|^2$) that the system (7.28) is asymptotically stable and

$$\|u(t)\| \leq \|u(0)\| e^{-\lambda t}.$$

Following Coron and Nguyen (2017) we also use the backstepping transformation in order to design a finite-time control of the form (7.26) with switching parameter λ. The system (7.28) will be utilized as a sort of comparison system in order to prove stability and finite-time convergence of solution to zero as well as to estimate the settling time.

One can be shown (Coron and Nguyen, 2017) that if $z \in L^2((0,1),\mathbb{R})$ then

$$\|z+F(\lambda)z\| \leq \Psi_1(\lambda)\|z\| \text{ and } \|z+F(-\lambda)z\| \leq \Psi_{-1}(\lambda)\|z\|,$$

$$\Psi_1(\lambda) = 1 + \frac{\lambda\sqrt{\int_0^1\int_0^x y^2\left(I_0\left(\sqrt{\lambda(x^2-y^2)}\right)-I_2\left(\sqrt{\lambda(x^2-y^2)}\right)\right)^2 dydx}}{2},$$

$$\Psi_{-1}(\lambda) = 1 + \frac{\lambda\sqrt{\int_0^1\int_0^x y^2\left(J_0\left(\sqrt{\lambda(x^2-y^2)}\right)+J_2\left(\sqrt{\lambda(x^2-y^2)}\right)\right)^2 dydx}}{2},$$

where J_k is the Bessel function of the first kind.

Now let us consider the heat system (7.22), (7.23) with the hybrid boundary feedback control

$$\xi(t) = \int_0^1 k\left(1, y, 2^{\sigma(t)}\right) v(t,y)dy, \tag{7.29}$$

where the switching function σ is governed by the equation (7.25) with

$$G(\sigma, v) = \begin{cases} i+1 & \text{if } \sigma = i \text{ and } \|v\| \leq r_{i+1}, \\ i & \text{if } \sigma = i \text{ and } r_{i+1} < \|v\| < r_{i-1}, \\ i-1 & \text{if } \sigma = i \text{ and } \|v\| \geq r_{i-1}, \end{cases} \tag{7.30}$$

where $r_0 = 1, r_i = e^{-q_i} r_{i-1}, i \in \mathbb{Z}$ and the numbers q_i are defined by formula (7.31).

In Polyakov *et al.* (2017) it is shown that

$$q_i = \ln \Psi_1\left(2^i\right) + \ln \Psi_{-1}\left(2^i\right), \tag{7.31}$$

then $q_i > 0$ for all $i \in \mathbb{Z}$, $r_i \to 0$ as $i \to +\infty$ and

$$\lim_{i\to+\infty} \frac{q_{i+1}}{q_i} = \sqrt{2}.$$

Moreover, for any initial condition

$$v(0) = v_0 \in L^2((0,1),\mathbb{R}) \tag{7.32}$$

and

$$\sigma(0) = i_0 \in \mathbb{Z} \quad \text{with} \quad \|v(0)\| \in \left(r_{i_0+1}, r_{i_0}\right], \tag{7.33}$$

the system (7.22), (7.23), (7.29), (7.25), (7.30) has a unique solution $(v, \sigma) : v \in C^0([0, T), L^2((0, 1), \mathbb{R}))$ and $\sigma : (0, T) \to \mathbb{Z}$ such that

$$\|v(t)\| \to 0 \text{ as } t \to T^-,$$

where $T = T(u_0)$ satisfies

$$T \leq \sum_{i=i_0}^{+\infty} \frac{q_i + q_{i+1}}{2^i} < +\infty. \tag{7.34}$$

Examples of the prescribed-time (*i.e.*, $T(u_0) \equiv$const) stabilization of PDEs can be found in Espitia *et al.* (2019) and Steeves *et al.* (2020).

Appendix

Notation

- $\mathbb{N}$ is the set of natural numbers; $\mathbb{Z}$ is the set of integers; $\mathbb{R}$ is the set of reals; $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ and $\mathbb{R}_+ = [0, +\infty)$; $\mathbb{C}$ is the set of complex numbers.
- $A \times B$ denotes the Cartesian product of sets A and B.
- The inner product of vectors $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ from a n-dimensional Euclidean space is given by $x \cdot y = \sum_{i=1}^{n} x_i y_i$, where x_i and y_i are coordinates of the vectors x and y in an orthonormal basis.
- $\mathbb{B}$ is a real Banach space with a norm $\|\cdot\|$ and $\mathbb{H}$ is a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$.
- $\mathbf{0} \in \mathbb{B}$ denotes a zero element a Banach space.
- $\text{span}\{e_1, e_2, ..., e_k\} := \{\alpha_1 e_1 + ... + \alpha_k e_k : \alpha_i \in \mathbb{R}, i = 1, 2, ..., k\}$, where $e_i \in \mathbb{B}$.
- $S = \{x \in \mathbb{B} : \|x\| = 1\}$ is the unit sphere in $\mathbb{B}$.
- The notation $\|\cdot\|_X$ is utilized if it is necessary to indicate that this is a norm in a space X.
- $\mathcal{L}(X, Y)$ is the space of linear bounded operators $X \to Y$, where X and Y are Banach spaces, and

$$\|A\|_{\mathcal{L}(X,Y)} = \sup_{u \neq 0} \frac{\|Au\|_Y}{\|u\|_X}, \quad \lfloor A \rfloor_{\mathcal{L}(X,Y)} = \inf_{u \neq 0} \frac{\|Au\|_Y}{\|u\|_X}, \quad A \in \mathcal{L}(X, Y).$$

We also use the notations $\|A\|$ and $\lfloor A \rfloor$ for shortness if a context is clear.

- $f_1(f_2)$ and $f_1 \circ f_2$ denote the composition of nonlinear operators (functions) f_1 and f_2. In the case of linear operators A and B, for simplicity, the brackets and the sign "$\circ$" can be omitted, *i.e.*, AB denotes the composition of linear operators A and B.

- If $P = P^\top \in \mathbb{R}^{n\times n}$ then $P \succ 0$ (resp. $\succeq 0$) means that the matrix P is positive definite (resp. semidefinite) and $P \prec 0$ (resp. $\preceq 0$) means that the matrix P is negative definite (resp. semidefinite).

- $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denotes minimum and maximum eigenvalues of a symmetric matrix $P = P^\top \in \mathbb{R}^n$.

- $\mathrm{rank}(A)$ denotes the rank of $A \in \mathbb{R}^{m\times n}$.

- $\mathrm{tr}(A)$ denotes the trace of $A \in \mathbb{R}^{n\times n}$.

- $G_{\mathbf{d}}^{-\top} := (G_{\mathbf{d}}^{-1})^\top$.

- $I \in \mathcal{L}(\mathbb{B},\mathbb{B})$ denotes the identity operator in $\mathbb{B}$ and I_n is the identity matrix in $\mathbb{R}^{n\times n}$.

- $\mathrm{div}(u) = \sum_{i=1}^{n} \frac{\partial u_i}{\partial z_i}$ for a function $u : \mathbb{R}^n \to \mathbb{R}^n$

- $\nabla := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$, and $\Delta := \nabla\cdot\nabla = \mathrm{div}(\nabla) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ - Laplace operator.

- $\overline{X}$ denotes the closure of the set X of a metric space.

- $C(X,Y)$ is the space of uniformly continuous functions $X \to Y$ with the supremum norm, $\|f\| = \sup_{x\in X}\|f(x)\|_Y$, where $f \in C(X,Y)$ and X, Y are normed vector spaces.

- $C_c^\infty(\Omega,\mathbb{R}^m)$ is the space of infinitely differentiable (smooth) functions having compact support in Ω, where $\Omega \subset \mathbb{R}^m$ is an open set.

- $C_0^\infty(\Omega, \mathbb{R}^m)$ is the space of infinitely differentiable (smooth) functions vanishing on the boundary of Ω.

- Let $L^p(\Omega, \mathbb{R}^m)$ denotes the Banach space of functions $\Omega \to \mathbb{R}^m$

$$L^p(\Omega, \mathbb{R}^m) := \{u : \|u\|_p < +\infty\},$$

$$\|u\|_{L^p} := \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p}, \quad 1 \leq p < \infty$$

$$\|u\|_{L^\infty} := \text{ess sup}|u(x)|, \quad p = \infty$$

- $L^2(\Omega, \mathbb{R}^m)$ is a Hilbert space with the inner product

$$\langle u, v \rangle_{L^2} = \int_\Omega u(x) \cdot v(x) \, dx.$$

- The Sobolev space $H^p(\Omega, \mathbb{R}^m)$ is a Hilbert space with the inner product

$$\langle u, v \rangle_{H^p} := \sum_{i=0}^{p} \langle \nabla^i u, \nabla^i v \rangle_{L^2}$$

and the norm $\| \cdot \|_{H^p} = \sqrt{\langle \cdot, \cdot \rangle_{H^p}}$.

- $H_0^p(\Omega, \mathbb{R}^m)$ is the completion of $C_c^\infty(\Omega, \mathbb{R}^m)$ with respect to $\| \cdot \|_{H^p}$.

- If $\Omega_1 \subset \mathbb{B}$ and $\Omega_2 \subset \mathbb{B}$ then, by definition, the identity $\Omega_1 = \Omega_2$ means $\Omega_1 \subset \Omega_2$ and $\Omega_2 \subset \Omega_1$.

- $B(r) := \{x \in \mathbb{B} : \|x\| < r\}$ is the open ball in $\mathbb{B}$ of the radius $r \in \mathbb{R}_+$ with the center at the origin and $B(y, r) = y + B(\varepsilon)$ is an open ball of the radius $\varepsilon > 0$ centered at $y \in \mathbb{B}$.

- $\partial\Omega$ is the boundary of a set $\Omega \subset \mathbb{R}^n$.

- $\text{int}(\Omega)$ denotes the interior of a set $\Omega \subset \mathbb{R}^n$, *i.e.*, $x \in \text{int}(\Omega)$ if and only if $\exists r \in \mathbb{R}_+ : x + \mathbb{B}(r) \subset \Omega$.

- The set consisting of elements $x_1, x_2, ..., x_n$ is denoted $\{x_1, x_2, ..., x_n\}$.

- The power set (*i.e.*, the set of all subsets) of a set $M \subset \mathbb{R}^n$ is 2^M.

- The sign function is defined by

$$\mathrm{sign}_\sigma(\rho) := \begin{cases} 1 & \text{if} \quad \rho > 0, \\ -1 & \text{if} \quad \rho < 0, \\ \sigma & \text{if} \quad \rho = 0, \end{cases} \qquad \text{(Appendix.1)}$$

  where $\sigma \in \mathbb{R} : -1 \leq \sigma \leq 1$. If a concrete value of σ is not important for considerations, we use the notation $\mathrm{sign}[\rho]$.

- $\lfloor x \rceil^\alpha := |x|^\alpha \, \mathrm{sign}[x]$ is the power operation, which preserves the sign of the number $x \in \mathbb{R}$. For example, $\lfloor -2 \rceil^2 = -4$ and $\lfloor 2 \rceil^2 = 4$.

- a function $\sigma \in C(\mathbb{R}_+, \mathbb{R}_+)$ belongs to the class $\mathcal{K}$ if $\sigma(0) = 0$ and σ is increasing, *i.e.*, $t_1 \leq t_2 \Rightarrow \sigma(t_1) \leq \sigma(t_2)$.

- a function $\sigma \in \mathcal{K}$ belongs to the class $\mathcal{K}^\infty$ if $\sigma(t) \to +\infty$ as $t \to +\infty$.

- a continuous function $\xi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\xi(\cdot, t) \in \mathcal{K}$ for any fixed $t \geq 0$ and $\xi(s, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing to zero for any fixed $s \geq 0$.

- $\mathcal{H}_{\mathbf{d}}(\mathbb{B})$ is a set of $\mathbf{d}$-homogeneous functionals $\mathbb{B} \to \mathbb{R}$ and $\deg_{\mathbf{d}}(h) \in \mathbb{R}$ is a homogeneity degree of $h \in \mathcal{H}_{\mathbf{d}}(\mathbb{B})$.

- $\mathcal{F}_{\mathbf{d}}(\mathbb{B})$ is a set of $\mathbf{d}$-homogeneous functionals $\mathbb{B} \to \mathbb{R}$ and $\deg_{\mathbf{d}}(f) \in \mathbb{R}$ is a homogeneity degree of $f \in \mathcal{F}_{\mathbf{d}}(\mathbb{B})$.

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