The Design of Competitive Online Algorithms via a Primal–Dual Approach
The Design of Competitive Online Algorithms via a Primal–Dual Approach

Niv Buchbinder

Technion — Israel Institute of Technology
Israel
nivb@cs.technion.ac.il

Joseph (Seffi) Naor

Technion — Israel Institute of Technology
Israel
naor@cs.technion.ac.il

Full text available at: http://dx.doi.org/10.1561/0400000024
Foundations and Trends® in
Theoretical Computer Science
Volume 3 Issue 2–3, 2007
Editorial Board

Editor-in-Chief:
Madhu Sudan
Department of CS and EE
MIT, Stata Center, Room G640
32 Vassar Street,
Cambridge MA 02139,
USA
madhu@mit.edu

Editors
Bernard Chazelle (Princeton)
Oded Goldreich (Weizmann Inst.)
Shafi Goldwasser (MIT and Weizmann Inst.)
Jon Kleinberg (Cornell University)
László Lovász (Microsoft Research)
Christos Papadimitriou (UC. Berkeley)
Prabhakar Raghavan (Yahoo! Research)
Peter Shor (MIT)
Madhu Sudan (MIT)
Éva Tardos (Cornell University)
Avi Wigderson (IAS)

Full text available at: http://dx.doi.org/10.1561/0400000024
Editorial Scope

Foundations and Trends® in Theoretical Computer Science will publish survey and tutorial articles in the following topics:

- Algorithmic game theory
- Computational algebra
- Computational aspects of combinatorics and graph theory
- Computational aspects of communication
- Computational biology
- Computational complexity
- Computational geometry
- Computational learning
- Computational Models and Complexity
- Computational Number Theory
- Cryptography and information security
- Data structures
- Database theory
- Design and analysis of algorithms
- Distributed computing
- Information retrieval
- Operations Research
- Parallel algorithms
- Quantum Computation
- Randomness in Computation

Information for Librarians

Foundations and Trends® in Theoretical Computer Science, 2007, Volume 3, 4 issues. ISSN paper version 1551-305X. ISSN online version 1551-3068. Also available as a combined paper and online subscription.
The Design of Competitive Online Algorithms via a Primal–Dual Approach

Niv Buchbinder\textsuperscript{1} and Joseph (Seffi) Naor\textsuperscript{2}

\textsuperscript{1} Computer Science Department, Technion — Israel Institute of Technology, Israel, nivb@cs.technion.ac.il
\textsuperscript{2} Computer Science Department, Technion — Israel Institute of Technology, Israel, naor@cs.technion.ac.il

Abstract

The primal–dual method is a powerful algorithmic technique that has proved to be extremely useful for a wide variety of problems in the area of approximation algorithms for NP-hard problems. The method has its origins in the realm of exact algorithms, e.g., for matching and network flow. In the area of approximation algorithms, the primal–dual method has emerged as an important unifying design methodology, starting from the seminal work of Goemans and Williamson [60].

We show in this survey how to extend the primal–dual method to the setting of online algorithms, and show its applicability to a wide variety of fundamental problems. Among the online problems that we consider here are the weighted caching problem, generalized caching, the set-cover problem, several graph optimization problems, routing, load balancing, and the problem of allocating ad-auctions. We also show that classic online problems such as the ski rental problem and the dynamic TCP-acknowledgement problem can be solved optimally using a simple primal–dual approach.
The primal–dual method has several advantages over existing methods. First, it provides a general recipe for the design and analysis of online algorithms. The linear programming formulation helps detecting the difficulties of the online problem, and the analysis of the competitive ratio is direct, without a potential function appearing “out of nowhere.” Finally, since the analysis is done via duality, the competitiveness of the online algorithm is with respect to an optimal fractional solution, which can be advantageous in certain scenarios.
# Contents

1 Introduction 1

2 Necessary Background 3
  2.1 Linear Programming and Duality 3
  2.2 Approximation Algorithms 7
  2.3 Online Computation 13
  2.4 Notes 15

3 A First Glimpse: The Ski Rental Problem 17
  3.1 Notes 21

4 The Basic Approach 23
  4.1 The Online Packing–Covering Framework 23
  4.2 Three Simple Algorithms 25
  4.3 Lower Bounds 35
  4.4 Two Warm-Up Problems 37
  4.5 Notes 41

5 The Online Set-Cover Problem 43
  5.1 Obtaining a Deterministic Algorithm 44
  5.2 Notes 48
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>The Metrical Task System Problem on a Weighted Star</td>
<td>51</td>
</tr>
<tr>
<td>6.1</td>
<td>A Modified Model</td>
<td>52</td>
</tr>
<tr>
<td>6.2</td>
<td>The Algorithm</td>
<td>55</td>
</tr>
<tr>
<td>6.3</td>
<td>Notes</td>
<td>56</td>
</tr>
<tr>
<td>7</td>
<td>Generalized Caching</td>
<td>59</td>
</tr>
<tr>
<td>7.1</td>
<td>The Fractional Weighted Caching Problem</td>
<td>60</td>
</tr>
<tr>
<td>7.2</td>
<td>Randomized Online Algorithm for Weighted Caching</td>
<td>71</td>
</tr>
<tr>
<td>7.3</td>
<td>The Generalized Caching Problem</td>
<td>76</td>
</tr>
<tr>
<td>7.4</td>
<td>Rounding the Fractional Solution Online</td>
<td>84</td>
</tr>
<tr>
<td>7.5</td>
<td>Notes</td>
<td>99</td>
</tr>
<tr>
<td>8</td>
<td>Load Balancing on Unrelated Machines</td>
<td>103</td>
</tr>
<tr>
<td>8.1</td>
<td>LP Formulation and Algorithm</td>
<td>103</td>
</tr>
<tr>
<td>8.2</td>
<td>Notes</td>
<td>106</td>
</tr>
<tr>
<td>9</td>
<td>Routing</td>
<td>109</td>
</tr>
<tr>
<td>9.1</td>
<td>A Generic Routing Algorithm</td>
<td>112</td>
</tr>
<tr>
<td>9.2</td>
<td>Achieving Coordinate-Wise Competitive Allocation</td>
<td>117</td>
</tr>
<tr>
<td>9.3</td>
<td>Notes</td>
<td>120</td>
</tr>
<tr>
<td>10</td>
<td>Maximizing Ad-Auctions Revenue</td>
<td>121</td>
</tr>
<tr>
<td>10.1</td>
<td>The Basic Algorithm</td>
<td>122</td>
</tr>
<tr>
<td>10.2</td>
<td>Multiple Slots: The Role of Strong Duality</td>
<td>125</td>
</tr>
<tr>
<td>10.3</td>
<td>Incorporating Stochastic information</td>
<td>129</td>
</tr>
<tr>
<td>10.4</td>
<td>Notes</td>
<td>134</td>
</tr>
<tr>
<td>11</td>
<td>Graph Optimization Problems</td>
<td>135</td>
</tr>
<tr>
<td>11.1</td>
<td>Formulating the Problem</td>
<td>136</td>
</tr>
<tr>
<td>11.2</td>
<td>The Group Steiner Problem on Trees</td>
<td>139</td>
</tr>
<tr>
<td>11.3</td>
<td>Notes</td>
<td>142</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>12 Dynamic TCP-Acknowledgement Problem</td>
<td>145</td>
<td></td>
</tr>
<tr>
<td>12.1 The Algorithm</td>
<td>146</td>
<td></td>
</tr>
<tr>
<td>12.2 Notes</td>
<td>148</td>
<td></td>
</tr>
<tr>
<td>13 The Bounded Allocation Problem: Beating (1 – 1/e)</td>
<td>149</td>
<td></td>
</tr>
<tr>
<td>13.1 The Algorithm</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>13.2 Notes</td>
<td>157</td>
<td></td>
</tr>
<tr>
<td>14 Extension to General Packing–Covering Constraints</td>
<td>159</td>
<td></td>
</tr>
<tr>
<td>14.1 The General Online Fractional Packing Problem</td>
<td>160</td>
<td></td>
</tr>
<tr>
<td>14.2 The General Online Fractional Covering Problem</td>
<td>165</td>
<td></td>
</tr>
<tr>
<td>14.3 Notes</td>
<td>168</td>
<td></td>
</tr>
<tr>
<td>15 Conclusions and Further Research</td>
<td>169</td>
<td></td>
</tr>
<tr>
<td>References</td>
<td>171</td>
<td></td>
</tr>
</tbody>
</table>
The primal–dual method is a powerful algorithmic technique that has proved to be extremely useful for a wide variety of problems in the area of approximation algorithms. The method has its origins in the realm of exact algorithms, e.g., for matching and network flow. In the area of approximation algorithms, the primal–dual method has emerged as an important unifying design methodology starting from the seminal work of Goemans and Williamson [60].

Our goal in this survey is to extend the primal–dual method to the setting of online algorithms, and show that it is applicable to a wide variety of problems. The approach we propose has several advantages over existing methods:

- A general recipe for the design and analysis of online algorithms is developed.
- The framework is shown to be applicable to a wide range of fundamental online problems.
- A linear programming formulation helps detecting the difficulties of the online problem in hand.
- The competitive ratio analysis is direct, without a potential function appearing “out of nowhere.”
Introduction

- The competitiveness of the online algorithm is with respect to an optimal fractional solution.

In Section 2, we briefly provide the necessary background needed for the rest of the discussion. This includes a short exposition on linear programming, duality, offline approximation methods, and basic definitions of online computation. Many readers may already be familiar with these basic definitions and techniques; however, we advise the readers not to skip this chapter, and in particular the part on approximation algorithms. Techniques pertinent to approximation algorithms are presented in a way that allows the reader to later see the similarity to the online techniques we develop. This section also provides some of the basic notation that we use in the sequel. Section 3 gives a first taste of the primal–dual approach in the context of online algorithms via the well-understood ski rental problem. We show an alternative way of deriving optimal algorithms for the ski rental problem using a simple primal–dual approach. In Section 4, we lay the foundations for the online primal–dual approach and design the basic algorithms for the packing–covering framework. We also study two toy examples that demonstrate the online framework. The rest of the sections show how to apply the primal–dual approach to many interesting and fundamental problems. We tried to make the chapters independent of each other; however, there are still certain connections between chapters, and thus closely related problems appear in consecutive chapters and typically in increasing order of complexity.

Among the problems that we consider are the weighted caching problem, generalized caching, the online set-cover problem, several graph optimization problems, routing, load balancing, and even the problem of allocating ad-auctions. We also show that classic online problems like the dynamic TCP-acknowledgement problem can be optimally solved using a primal–dual approach. There are also several more problems that can be solved via the primal–dual approach and are not discussed here. Such problems are, for example, the admission control problem [5], the parking permit problem [83] and the inventory problem [31].
References


References


References


References


References


