On Doubly-Efficient Interactive Proof Systems

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On Doubly-Efficient Interactive Proof Systems

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Abstract

An interactive proof system is called doubly-efficient if the prescribed prover strategy can be implemented in polynomial-time and the verifier’s strategy can be implemented in almost-linear-time. Such proof systems, introduced by Goldwasser, Kalai, and Rothblum (JACM, 2015), make the benefits of interactive proof system available to real-life agents who are restricted to polynomial-time computation.

We survey some of the known results regarding doubly-efficient interactive proof system. We start by presenting two simple constructions for \textit{t}-no-\textsc{CLIQUE} (due to Goldreich and Rothblum (ECCC, TR17-018 and TR18-046)), where the first construction offers the benefit of being generalized to any “locally characterizable” set, and the second construction offers the benefit of preserving the combinatorial flavor of the problem. We then turn to two more general constructions of doubly-efficient interactive proof system: the proof system for sets having (uniform) bounded-depth circuits (due to Goldwasser, Kalai and Rothblum (JACM, 2015)), and the proof system for sets that are recognized in polynomial-time and small space (due to Reingold, Rothblum, and Rothblum (STOC, 2016)). Our presentation of the GKR construction is quite complete (and is somewhat different from the original presentation), but for the RRR construction we only provide an overview.

The notion of interactive proof systems, put forward by Goldwasser, Micali, and Rackoff [28], and the demonstration of their power by Lund, Fortnow, Karloff, and Nisan [35] and Shamir [44] are among the most celebrated achievements of complexity theory. Recall that an interactive proof system for a set $S$ is associated with an interactive verification procedure, $V$, that can be made to accept any input in $S$ but no input outside of $S$. That is, there exists an interactive strategy for the prover that makes $V$ always accept any input in $S$ but no strategy can make $V$ accept an input outside of $S$, except with negligible probability. (See Appendix A.1 for a formal definition of interactive proofs, and [19, Chap. 9] for a wider perspective.)

The original definition does not restrict the complexity of the strategy of the prescribed prover and the constructions of [35, 44] use prover strategies of high complexity. This fact limits the applicability of these proof systems in practice. (Nevertheless, such proof systems may be actually applied when the prover knows something that the verifier does not know, such as an NP-witness to an NP-claim; this is beneficial when the proof system offers an advantage (over NP-proof systems) such as being zero-knowledge [28, 22].)
1.1 The notion of doubly-efficient interactive proof systems

Seeking to make interactive proof systems available for a wider range of applications, Goldwasser, Kalai and Rothblum put forward a notion of doubly-efficient interactive proof systems (also called interactive proofs for muggles [27] and interactive proofs for delegating computation [42]). In these proof systems the prescribed prover strategy can be implemented in polynomial-time and the verifier’s strategy can be implemented in almost-linear-time. That is, doubly-efficient interactive proof systems are restricted by two additional efficiency requirements:

*Prover’s efficiency requirement:* The prescribed prover strategy (referred to in the completeness condition) should be implemented in polynomial-time.

*Verifier’s efficiency requirement:* The verifier strategy should be implemented in almost-linear time.

(We stress that unlike in argument systems, the soundness condition holds for all possible cheating strategies (not only for feasible ones).)\(^1\)

Restricting the prescribed prover to run in polynomial-time implies that such systems may exist only for sets in \(\mathbf{BPP}\), whereas a polynomial-time verifier can check membership in such sets by itself. However, restricting the verifier to run in almost-linear-time implies that something can be gained by interacting with a more powerful prover, even though the latter is restricted to polynomial-time.

The potential applicability of doubly-efficient interactive proof systems was demonstrated by Goldwasser, Kalai and Rothblum [27], who constructed such proof systems for any set that has log-space uniform circuits of small depth (e.g., log-space uniform \(\mathbf{NC}\)). A recent work of Reingold, Rothblum, and Rothblum [42] provided doubly-efficient (constant-round) proof systems for any set that can be decided in polynomial-time and small amount of space (e.g., for all sets in \(\mathbf{SC}\)). These two results are actually incomparable. The two constructions will be reviewed in Chapters 3 and 4, respectively, but before doing so we shall consider simpler constructions (see Section 1.2 and Chapter 2).

\(^1\)See further discussion in Section 1.5.
**Terminology:** We keep the term *almost linear* vague on purpose, but whenever appropriate we shall spell-out a specific interpretation of it. The most strict interpretation is that a function \( f : \mathbb{N} \to \mathbb{N} \) is *almost linear* if \( f(n) = \tilde{O}(n) = \text{poly}(\log n) \cdot n \). This interpretation is suitable for Chapter 3 and most of Chapter 2 (i.e., for the results of [27] and [24, 25], resp). In contrast, Chapter 4 (following [42]) uses a much more liberal interpretation by which a sequence of functions of the form \( f_\epsilon : \mathbb{N} \to \mathbb{N} \) (representing the complexities of a sequence of constructions) is *almost linear* if \( f_\epsilon(n) = O(n^{1+\epsilon}) \) for every \( \epsilon > 0 \). We mention that these interpretations have been used in the past also in other settings (see discussion in [20, Sec. 13.3.3]).

### 1.2 Doubly-efficient NP-proof systems

For starters, we mention that doubly-efficient interactive proof systems exist for some sets in \( \mathcal{P} \) that are believed not to have almost-linear decision procedures (or at least are not known to have such procedures). Examples include perfect matching and \( t\text{-CLIQUE} \), for any constant \( t \geq 3 \). In these cases, the proof systems are actually of the NP-type. The point is that in each of these cases, an easily verified NP-witness (i.e., one that can be verified in almost-linear time) can be found in polynomial-time.

The foregoing assertion is quite evident for perfect matching, by virtue of the known matching algorithms and the fact that checking whether a set of edges is a perfect matching in a given graph can be done in linear time. The same hold for the maximum flow problem, by virtue of the min-cut dual. More generally, morally, the same holds for linear programming (again by duality; i.e., by checking the feasibility of the dual program).\(^2\) Turning to fixed-parameter problems, such as \( t\text{-CLIQUE} \), let us now consider the (popular) sets

\( t\text{-CLIQUE} \): The set of \( n \)-vertex graphs that contain a clique of size \( t \).

\(^2\)Formally, issues may arise with respect to the length of the description of the solutions to the primal and dual programs, but these issues can be resolved by padding the input to that length (which seems quite natural in this context).
1.2. Doubly-efficient NP-proof systems

**t-SUM:** The set of $n$-long sequences of integers, say, in $[-n^{t+3}, n^{t+3}]$, that contain $t$ elements that sum-up to zero.

**t-OV:** The set of $n$-long sequences of vectors over $\text{GF}(2)^{\log_2 n}$ that contain $t$ vectors such that their coordinate-wise multiplication yields the all-zero vector.

In all cases, the NP-witness is a set of $t$ elements, which can be found in time $\binom{n}{t}$ and verified in $\text{poly}(t \cdot \log n)$-time. Recall that $t$-CLIQUE is conjectured to require time $n^{c'_t}$, where $c'_t$ is any constant smaller than one third of the Boolean matrix multiplication exponent (see, e.g., [1]), and that 3-SUM is conjectured to require almost quadratic time (see, e.g., [40], which promoted it), whereas $t$-OV generalizes the Orthogonal Vectors problem (see [6]).

Furthermore, $t$-CLIQUE is $W[1]$-complete [15], solving it in time $n^{o(t)}$ refutes the ETH [13], and lower bounds for $t$-SUM are known to follow from lower bounds for $t$-CLIQUE (see [2]).

In general, the class of sets having doubly-efficient NP-proof systems is a subclass of $\mathcal{P} \cap \text{Ntime}(\tilde{L})$, where $\tilde{L}$ denotes the set of almost-linear functions. (Indeed, the class of sets having doubly-efficient NP-proof systems consists of all sets $S \in \text{Ntime}(\tilde{L})$ associated with a witness relation $R$ such that given $x \in S$ we can find $y \in R(x)$ in polynomial-time.)

Formally, $S$ has a doubly-efficient NP-proof system if and only if there exists a relation $R$ such that

1. if $(x, y) \in R$, then $|y|$ is almost linear in $|x|$;
2. $S = \{x : \exists w \text{ s.t. } (x, w) \in R\}$;
3. there exists a polynomial-time algorithm that, on input $x \in S$, outputs an element of $\{w : (x, w) \in R\}$;
4. there exists an almost-linear time algorithm that, on input $(x, y)$, decides whether or not $(x, y) \in R$.

\[^{\text{4}}\text{Indeed, the Orthogonal Vectors problem corresponds to the case of } t = 2. \text{ Our formulation of } t\text{-OV is different but equivalent to the one in [6], where the sequence of vectors is partitioned into } t \text{ equal parts and a } \text{yes-instance has to take a single vector from each part.}\]

\[^{\text{4}}\text{Formally, } S \text{ has a doubly-efficient NP-proof system if and only if there exists a relation } R \text{ such that}\]

1. if $(x, y) \in R$, then $|y|$ is almost linear in $|x|$;
2. $S = \{x : \exists w \text{ s.t. } (x, w) \in R\}$;
3. there exists a polynomial-time algorithm that, on input $x \in S$, outputs an element of $\{w : (x, w) \in R\}$;
4. there exists an almost-linear time algorithm that, on input $(x, y)$, decides whether or not $(x, y) \in R$.\]
beyond doubly-efficient NP-proof systems, we focus on the power of doubly-efficient interactive proof systems.

1.3 The power of doubly-efficient interactive proof systems

The bulk of this survey is devoted to demonstrating the power of doubly-efficient interactive proof systems. We shall start by presenting two different (doubly-efficient interactive) proof systems for \( t\text{-no-CLIQUE} \) (i.e., the complement of \( t\text{-CLIQUE} \)). These proof systems (presented in Sections 2.1 and 2.3) are considerably simpler than the proof systems that can be derived from the general results captured by Theorems 1.1 and 1.2 (and presented in Chapters 3 and 4).

Before turning to these more general results, we briefly discuss the two doubly-efficient interactive proof systems for \( t\text{-no-CLIQUE} \). One of these systems (i.e., the one presented in Section 2.3) proceeds in \( t \) rounds such that, in the \( i^{th} \) round, a claim regarding the number of \((t - i + 1)\text{-cliques}\) in a graph is reduced to a claim regarding the number of \((t - i)\text{-cliques}\) in a related graph. Hence, in each iteration, a natural computational problem is reduced to a closely related computational problem, while preserving the combinatorial flavor of the original problem. This proof system can also handle varying \( t \), yielding an alternative interactive proof system for \( \#P \).

The idea that underlies the other proof system for \( t\text{-no-CLIQUE} \) (presented in Section 2.1) can be applied to a natural class of “locally characterizable” sets (defined in Section 2.2, following [24, Sec. 5]). This class, which contains \( t\text{-no-CLIQUE} \), is a subclass of \( NC \cap SC \). This means that doubly-efficient interactive proof systems for locally characterizable sets can be obtained from either Theorem 1.1 or Theorem 1.2, but the point of presenting the direct proof systems for locally characterizable sets (in Section 2.2) is that they are considerably simpler than those obtained by either Theorem 1.1 or Theorem 1.2. Still, the latter theorems yield the most general results known regarding doubly-efficient interactive proof systems.

\(^5\)The two simple proof systems for \( t\text{-no-CLIQUE} \) are due to [24, Sec. 3] and [25, Sec. 2], resp., whereas Theorems 1.1 and 1.2 are due to [27] and [42], resp.
1.3. The power of doubly-efficient interactive proof systems

**Theorem 1.1** (doubly-efficient interactive proof systems for log-space uniform $\mathcal{NC}$ [27]). Every set that is decidable by a family of log-space uniform circuits of depth $d$ such that $d(n) = \tilde{O}(n)$, has a doubly-efficient interactive proof system. Furthermore, the proof system uses $O(\log n \cdot d(n))$ rounds, and the verifier runs in $\tilde{O}(n + d(n))$-time.

Although circuit size and depth is related to time and space [10], this relation is not tight enough to relate $\mathcal{NC}$ and $\mathcal{SC}$.\(^6\) Hence, Theorem 1.1 is incomparable to the following

**Theorem 1.2** (doubly-efficient interactive proof systems for $\mathcal{SC}$ [42]). Every set that is decidable by an algorithm that runs in polynomial time and has space complexity $s$ such that $s(n) < \sqrt{n}$, has a doubly-efficient interactive proof system. Furthermore, for any constant $\delta > 0$, the proof system uses $\exp(\tilde{O}(1/\delta))$ rounds, and verifier runs in $(\tilde{O}(n) + s(n)^2 \cdot n^\delta)$-time.

As noted in [42], Theorem 1.2 can be extended to randomized algorithms by first reducing their randomness complexity to linear (using adequate pseudorandom generators), and then letting the verifier toss coins for the derived algorithm and send the outcomes to the prover (asking it to prove membership in the corresponding residual set).\(^7\) A begging open problem is whether the upper bound of $s(n)^2$ can be replaced by $s(n)$; that is,

**Problem 1.3** (a possible quantitative improvement of Theorem 1.2). Does every set that is decidable by an algorithm that runs in polynomial time and linear space have a doubly-efficient interactive proof system?

\(^6\)The point is that the translations between depth and space do not preserve polynomial bounds on the size and time, respectively. Specifically, $\mathcal{SC}$ can be emulated by uniform circuits of polylogarithmic depth (and quasi-polynomial size), whereas log-space uniform $\mathcal{NC}$ can be emulated by algorithms of polylogarithmic space complexity (that run in quasi-polynomial time).

\(^7\)When applied to a randomized algorithm with two-sided error this yields an interactive proof system with two-sided error (a.k.a imperfect completeness (see Appendix A.1)). Recall, however, that we our focus is on interactive proof systems with perfect completeness (as in Definition A.1); such proof systems are obtained here when applying the foregoing reduction to a randomized algorithm that always accepts yes-instances. A similar comment holds with respect to all subsequent statements about $\mathcal{BPP}$ (see, e.g., Theorem 1.4 and Section 1.5); that is, when considering proof systems of perfect completeness, one may replace $\mathcal{BPP}$ by $\text{coRP}$.\(^8\)
Another intriguing question is whether the round complexity in Theorem 1.2 can be reduced to $\text{poly}(1/\delta)$. We mention that all the aforementioned proof systems, which will be surveyed in the subsequent chapters, are of the public-coin type.\footnote{In the public coin model, at each round, the verifier tosses a predetermined number of coins and sends the outcome to the prover (see discussion at the end of Appendix A.1).} Note that the known transformation of general interactive proof systems to public-coin ones does not apply to doubly-efficient interactive proof systems, since the resulting prover strategy is not efficient and this seems inherent [48]. This begs the question of whether general systems (i.e., ones that are not of the public-coin type) can offer some advantages in the context of doubly-efficient interactive proof systems.

1.4 An upper bound on doubly-efficient interactive proof systems

As stated upfront, doubly-efficient interactive proof systems exists only for sets in $\mathcal{BPP}$. This is the case, since a decision procedure can just emulate the interaction between the prescribed polynomial-time prover (and the polynomial-time verifier). Using the hypothesis that the verifier runs in almost linear time, it follows that such sets are decidable in almost linear space. This is the case since the hypothesis implies that the communication and the verifier’s randomness are (at most) almost-linear (in the length of the input), and the same holds for the space complexity of the verifier. Hence, a machine of almost linear space complexity can decide membership in the set by emulating all possible interactions. For future reference, let us state the conclusion of the foregoing discussion.

**Theorem 1.4 (upper bound).** Every set that has a doubly-efficient interactive proof system can be decided in $\mathcal{BPP} \cap \text{Dspace}(\tilde{\ell})$, where $\tilde{\ell}$ is an almost linear function.

Note that even if Theorem 1.2 is improved as suggested in Problem 1.3, a gap will remain between it and Theorem 1.4, since $\text{TiSp}(T, s)$ is not necessarily equal $\text{Time}(T) \cap \text{Space}(s)$. Hence, another begging
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open problem is whether the power of doubly-efficient interactive proof systems is captured by $\text{TiSp}(\text{poly}, s)$ or by $\mathcal{BPP} \cap \text{Space}(s)$ (or by neither).

**Problem 1.5** (on the gap between Theorems 1.2 and 1.4). Prove or provide evidence against at least one of the following conjectures:

1. For $s(n) = \tilde{O}(n)$, every set in $\mathcal{P} \cap \text{Dspace}(s)$ has a doubly-efficient interactive proof system. Even establishing the claim for some $s(n) = \omega(\log n)$ would be interesting.\(^9\)

2. Every set that has a doubly-efficient interactive proof system can be decided by a probabilistic algorithm that runs in polynomial time and almost linear space.

We mention that the second conjecture contradicts the conjecture that *log-space uniform circuits of linear depth and polynomial size cannot be emulated by polynomial-time algorithms of almost-linear space complexity*, since Theorem 1.1 provides doubly-efficient interactive proof systems for the former. Can stronger evidence be brought against the second conjecture? On the other hand, we propose a milder form of the second conjecture asserting that every set that has a constant-round doubly-efficient interactive proof system can be decided by a probabilistic algorithm that runs in polynomial time and almost linear space.

### 1.5 On doubly-efficient argument systems

Recall that argument systems are defined as interactive proof systems with the exception that the soundness condition is replaced by a computational soundness condition. That is, while the (standard) soundness condition requires that no cheating strategy can make the prover accept false assertions, except with negligible probability, the computational soundness condition only requires the infeasibility of cheating (i.e., that

\(^9\)Note that, by Theorem 1.1, the claim holds for any $s(n) = O(\log n)$, since $\mathcal{L}$ is contained in log-space uniform $\mathcal{NC}$.\)
cheating strategies that can be implemented by polynomial size circuits may only fool the verifier with negligible probability).\textsuperscript{10}

Doubly-efficient argument systems for any set in $\mathcal{BPP}$ are implicit in Kilian’s argument system for sets in $\mathcal{NP}$, which relies on collision resistance hashing \[32].\textsuperscript{11} This system uses a constant number of rounds. Furthermore, assuming the existence of computational PIR schemes \[33], every set in $\mathcal{BPP}$ has a two-message doubly-efficient argument system \[30, 31, 11].\textsuperscript{12} On the other hand, any set having a doubly-efficient argument system is in $\mathcal{BPP}$ (since we can decide membership in such a set by emulating the interaction between the prescribed prover and verifier strategies).\textsuperscript{13}

The fact that argument systems are always asserted by relying on an intractability assumption is no coincidence, since these asserted systems do not satisfy the information theoretic soundness requirement. In fact, the existence of an argument system that is not an interactive proof system (i.e., does not satisfy standard soundness) implies a complexity separation (which is not known unconditionally). Specifically:

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\textsuperscript{10}Polynomial size circuits are preferred over probabilistic polynomial-time algorithms in order to account for auxiliary information that may be available to the prover (esp., when used as a subroutine inside a higher-level application).

\textsuperscript{11}The claim is evident for sets in $\mathcal{P}$, whereas proving membership in $\mathcal{BPP}$-sets can be reduced to proving membership in $\mathcal{P}$-set as follows. First note that the hypothesis (i.e., the existence of collision resistance hashing) implies the existence of one-way functions, and hence of pseudorandom generators \[29]. Using such a generator, we can reduce the randomness complexity of the decision procedures for $\mathcal{BPP}$ to linear, and let the verifier send a random-tape for such a procedure (as part of its first message). Hence, it suffices to verify a claim that refers to the residual set, which is in $\mathcal{P}$.

\textsuperscript{12}The result of \[31], which builds on \[30], uses a computational PIR of quasi-polynomial security. The assumption was weakened to standard (polynomial security) by \[11]. The original results that are stated for $\mathcal{P}$ can be extended to $\mathcal{BPP}$ (cf. Footnote 11).

\textsuperscript{13}The validity of this decision procedure refers only to the probability that the prescribed prover convinces the (prescribed) verifier.
1.6 Preliminaries: The Sum-Check protocol

Theorem 1.6 (arguments that are not proofs imply separations). Let $V$ be a verifier strategy for an argument system for a set $S$, and suppose that $V$ does not satisfy the (information theoretic) soundness requirement. Then, $\mathcal{PSPACE}$ is not contained in $\mathcal{P}/\text{poly}$. In particular, $\mathcal{BPP} \neq \mathcal{PSPACE}$.$^{14}$

In other words, Theorem 1.6 asserts that a gap between information theoretic soundness and computational soundness means a gap between computationally unbounded (prover) strategies and computationally bounded (prover) strategies. Recalling that, in the current setting (of fooling a probabilistic polynomial-time verifier), the former can be implemented in $\mathcal{PSPACE}$, whereas the computational restriction refers to $\mathcal{P}/\text{poly}$, the main claim follows (and $\mathcal{BPP} \neq \mathcal{PSPACE}$ follows, since $\mathcal{BPP} \subseteq \mathcal{P}/\text{poly}$).

Proof: Assume, for simplicity and without loss of generality, that in the said argument system each message of the prover consists of a single bit, and let $f : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$ denote an optimal prover strategy with respect to this system (i.e., $f(x,\gamma)$ is the message sent by the prover on common input $x$ after receiving the sequence of verifier messages described in the communication transcript $\gamma$). Then, $f \in \mathcal{PSPACE}$, since an optimal prover strategy (w.r.t a probabilistic polynomial-time verifier) can be implemented in polynomial space. On the other hand, $f \notin \mathcal{P}/\text{poly}$, since otherwise a polynomial size circuit could implement the optimal strategy (for convincing $V$), and so there will be no gap between the information theoretic soundness and the computational soundness of this proof system. 

1.6 Preliminaries: The Sum-Check protocol

The Sum-Check protocol, designed by Lund, Fortnow, Karloff, and Nisan [35], is a key ingredient in some of the constructions that we present. In particular, it will used in Sections 2.1 and 2.2 as well as in Chapter 3.

$^{14}$Note that $\mathcal{BPP} \neq \mathcal{PSPACE}$ implies $\mathcal{BPP} \subseteq \mathcal{PSPACE}$, since $\mathcal{BPP} \subseteq \mathcal{PSPACE}$. 
Fixing a finite field $\mathcal{F}$ and a set $H \subset \mathcal{F}$ (e.g., $H$ may be a two-element set), we consider an $m$-variate polynomial $P : \mathcal{F}^m \to \mathcal{F}$ of individual degree $d$. Given a value $v$, the Sum-Check protocol is used to prove that

$$\sum_{\sigma_1, \ldots, \sigma_m \in H} P(\sigma_1, \ldots, \sigma_m) = v,$$

assuming that the verifier can evaluate $P$ by itself. The Sum-Check protocol proceeds in $m$ iterations, such that in the $i^{th}$ iteration the number of summations (over $H$) decreases from $m - i + 1$ to $m - i$. Specifically, the $i^{th}$ iteration starts with a claim of the form

$$\sum_{\sigma_i, \ldots, \sigma_m \in H} P(r_1, \ldots, r_{i-1}, \sigma_i, \ldots, \sigma_m) = v_{i-1},$$

where $r_1, \ldots, r_{i-1}$ and $v_{i-1}$ are as determined in prior iterations (with $v_0 = v$), and ends with a claim of the form

$$\sum_{\sigma_{i+1}, \ldots, \sigma_m \in H} P(r_1, \ldots, r_i, \sigma_{i+1}, \ldots, \sigma_m) = v_i,$$

where $r_i$ and $v_i$ are determined in the $i^{th}$ iteration. Initializing the process with $v_0 = v$, in the $i^{th}$ iteration the parties act as follows.

**Prover’s move:** The prover computes a univariate polynomial of degree $d$ over $\mathcal{F}$

$$P_i(z) \overset{\text{def}}{=} \sum_{\sigma_{i+1}, \ldots, \sigma_m \in H} P(r_1, \ldots, r_{i-1}, z, \sigma_{i+1}, \ldots, \sigma_m),$$

where $r_1, \ldots, r_{i-1}$ are as determined in prior iterations, and sends $P_i$ to the verifier (claiming that $\sum_{\sigma \in H} P_i(\sigma) = v_{i-1}$).

**Verifier’s move:** Upon receiving a degree $d$ polynomial, denoted $\tilde{P}$, the verifier checks that $\sum_{\sigma \in H} \tilde{P}(\sigma) = v_{i-1}$ and rejects if inequality holds. Otherwise, it selects $r_i$ uniformly in $\mathcal{F}$, and sends it to the prover, while setting $v_i \leftarrow \tilde{P}(r_i)$.

If all $m$ iterations are completed successfully (i.e., without the verifier rejecting in any of them), the verifier conducts a final check. It computes the value of $P(r_1, \ldots, r_m)$ and accepts if and only if this value equals $v_m$.

Clearly, if Eq. (1.1) holds (and the prover acts according to the protocol), then the verifier accepts with probability 1. Otherwise, no matter what the prover does, the verifier accepts with probability at most $m \cdot d/|\mathcal{F}|$, because in each iteration if the prover provides the correct
polynomial, then the verifier rejects (since \( \sum_{\sigma \in H} P_i(\sigma) = P_{i-1}(r_{i-1}) \neq v_{i-1} \)), and otherwise the (degree \( d \)) polynomial sent agrees with \( P_i \) on at most \( d \) points.\(^{15}\)

The complexity of verification is dominated by the complexity of evaluating \( P \) (on a single point). As for the prescribed prover, it may compute the relevant \( P_i \)'s by interpolation, which is based on computing the value of \( P \) at \( (d + 1) \cdot |H|^{m-i} \) points, for each \( i \in [m] \). (That is, the polynomial \( P_i \) is computed by obtaining its values at \( d + 1 \) points, where the value of \( P_i \) at each point is obtained by summing the values of \( P \) at \( |H|^{m-i} \) points.)\(^{16}\)

### 1.7 Organization

In Chapter 2 we present simple constructions of doubly-efficient interactive proof systems for \( t \)-no-\textsc{clique}, as well as for a natural class that contains it and is contained in uniform \( \mathcal{NC} \) (and also in \( \mathcal{SC} \)). These proof systems are due to Goldreich and Rothblum [24, 25]. In Chapter 3 we present the proof systems of Goldwasser, Kalai and Rothblum [27], which are applicable to sets that are recognized by small depth circuits (e.g., uniform \( \mathcal{NC} \)). In Chapter 4 we provide an outline of the proof systems of Reingold, Rothblum, and Rothblum [42], which are applicable to sets recognized in polynomial-time and small space (e.g., sets in \( \mathcal{SC} \)).

As hinted in the foregoing paragraph, Chapter 2 is much easier to read than Chapters 3 and 4. Furthermore, while Chapters 2 and 3

\(^{15}\)If \( P_i \) does not satisfy the current claim (i.e., \( \sum_{\sigma \in H} P_i(\sigma) \neq i \cdot v_{i-1} \)), then the prover can avoid upfront rejection only if it sends \( \bar{P} \neq P_i \). But in such a case, \( \bar{P} \) and \( P_i \) (both being degree \( d \) polynomials) may agree on at most \( d \) points. Hence, if the chosen \( r_i \in \mathcal{F} \) is not one of these points, it holds that \( v_i = \bar{P}(r_i) \neq P_i(r_i) \), which means that the next iteration will also start with a false claim. Hence, starting with a false claim (i.e., \( \sum_{\sigma \in H} P_i(\sigma) \neq i \cdot v_0 \) since Eq. (1.1) does not hold), with probability at least \( 1 - m \cdot d/|\mathcal{F}| \), after \( m \) iterations we reach a false claim regarding the value of \( P \) at a single point.

\(^{16}\)Specifically, the value of \( P_i \) at \( p \) is obtained from the values of \( P \) at the points \( (r_1, \ldots, r_{i-1}, p, \sigma) \), where \( \sigma \in H^{m-i} \).
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provide full expositions of the claimed proof systems, Chapter 4 provides only an overview of the claimed proof system (while referring the interested reader to the 70-page description in the original work [42]).

Each of the following three chapters starts with several overview paragraphs that outline the contents of the chapter. Furthermore, Chapters 3 and 4 proceed with overview sections (see Sections 3.1 and 4.1, respectively).

We conclude (in Chapter 5) with an attempt to provide a high-level digest of the four fundamentally different proof systems reviewed above and with some speculations regarding the study of interactive proof systems at large.

Conventions: We assume that the verifier (resp., prover) has direct access to the common input; that is, each bit in the input can be read in unit cost. Unless explicitly stated differently, all logarithms are to base 2.
References


References


References


References


