Informative positive and negative campaigning – Appendix

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1 Proof of Proposition 2

Let

$$NB = \{(\alpha, v_2) | v_2 \in [\underline{v}_2, \overline{v}_2] \wedge \alpha = \inf_{\tilde{w}} (\tilde{w}, v_2) \in \mathcal{P}\} \tag{1}$$

and

$$PB = \{(w, v_2) | v_2 \in [\underline{v}_2, \overline{v}_2] \wedge \alpha = \sup_{\tilde{w}} (\tilde{w}, v_2) \in \mathcal{N}\} \tag{2}$$

To the left of NB, all points are elements of $\mathcal{N}$, and to the right of PB, all points are elements of $\mathcal{P}$. If the claim of the proposition is false, then NB and PB do not coincide (see Figure 1 below).

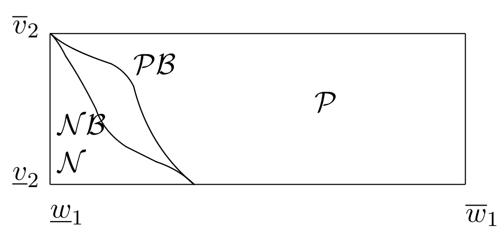

Figure 1: Parameter space with boundary lines $\mathcal{PB}$ and $\mathcal{NB}$

Pick a point $(w_1^0, v_2^0) \in \mathcal{P}$ on (or within ε of) NB and strictly to the left of PB. Moreover, there exists $(w_1', v_2^0) \in \mathcal{N}$ within ε of PB, so that $w_1' \geq w_1^0$.

Since candidate 1 informs on w_1^0 at (w_1^0, v_2^0), it must be true that

$$w_1^0 - E(v_2|w_1^0) \geq E(w_1|v_2^0) - v_2^0 \tag{3}$$

On the other hand, by monotonicity, the equilibrium utility at $(w_1', v_2^0) \in \mathcal{N}$ must be at least as large as the equilibrium utility at (w_1^0, v_2^0):

$$E(w_1|v_2^0) - v_2^0 \geq w_1^0 - E(v_2|w_1^0). \tag{4}$$

Hence, the equilibrium utility is the same at (w_1^0, v_2^0) and at (w_1', v_2^0):

$$w_1^0 - E(v_2|w_1^0) = E(w_1'|v_2^0) - v_2^0. \tag{5}$$

Moreover, all points (w_1'', v_2^0) with $w_1'' \in (w_1^0, w_1']$ must be elements of $\mathcal{N}$. (Suppose there was a point $(w_1'', v_2^0) \in \mathcal{P}$; then candidate 1's equilibrium utility at this point would be higher than at (w_1^0, v_2^0), by the monotone equilibrium assumption: $w_1'' - E(v_2|w_1'') > w_1^0 - E(v_2|w_1^0)$. However, using (5), this implies that equilibrium utility at (w_1'', v_2^0) is larger than equilibrium utility at (w_1', v_2^0), $w_1'' - E(v_2|w_1'') > E(w_1'|v_2^0) - v_2^0$, which contradicts the monotone equilibrium assumption.)

Now, take any point $(\tilde{w}_1, \tilde{v}_2) \in \mathcal{N}$ within ε of PB. By a similar argument as above, all points $(\tilde{w}_1, v_2'')$ with $v_2'' \in [v_2', \tilde{v}_2)$ must be elements of $\mathcal{P}$. Hence, all points that are strictly between PB and NB must be elements of both $\mathcal{P}$ and $\mathcal{N}$, the desired contradiction. $\square$

2 Proof of Proposition 5

Case 1 expected utility. We first calculate the voter's expected utility in the equilibrium of Proposition 3. To do this, we calculate the expected quality of candidate 1, given that he wins. Since candidates are ex-ante symmetric, this is also the expected quality of the winning candidate. Suppose first that $\overline{v} - \underline{v} \geq 1 (= \overline{w} - \underline{w})$.

For a given $(w_1, v_2) \in \mathcal{N}$ where $v_2 \leq \overline{v} - 1$, Candidate 1 wins if and only if $1/2 - v_2 \geq \max(1/2 - v_1, w_2 - (\overline{v} - \frac{1}{2}w_2))$. The left hand side is the voters expectation of $w_1 - E(v_2)$, while the right hand side is the optimal $E(w_2 - v_1)$ that Candidate 2 can achieve. If $v_2 \leq \overline{v} - 1$, Candidate 2 can only win if he uses negative campaigning and $v_1 < v_2$, since with positive campaigning by Candidate 2, $1/2 - v_1$ is always larger than $(3/2)w_1 - \overline{v}$. Hence, in this case, the probability that A wins for $(w_1, v_2) \in \mathcal{N}$ where $v_2 \leq \overline{v} - 1$ is

$$\Pr(A|(w_1, v_2) \text{ s.t. } v_2 \leq \overline{v} - 1) = \frac{\overline{v} - v_2}{\overline{v}} \tag{6}$$

The conditional expectation of Candidate 1's quality in this case is

$$E(w_1 + v_1|(w_1, v_2) \text{ s.t. } v_2 \leq \overline{v} - 1 \text{ and C1 wins}) = w_1 + \frac{v_2 + \overline{v}}{2} \tag{7}$$

Consider now $(w_1, v_2) \in \mathcal{N}$ where $v_2 \geq \overline{v} - 1$. Candidate 1 wins if and only if both $v_1 \geq v_2$ and $w_2 \leq (\overline{v} - v_2)$, hence with probability

$$\Pr(A|(w_1, v_2) \text{ s.t. } v_2 \in (\overline{v} - 1, \overline{v}) \text{ and C1 wins}) = \frac{(\overline{v} - v_2)^2}{\overline{v}} \tag{8}$$

The conditional expectation of Candidate 1's quality in this case is

$$E(w_1 + v_1|(w_1, v_2) \text{ s.t. } v_2 \in (\overline{v} - 1, \overline{v})) \text{ and C1 wins}) = w_1 + \frac{v_2 + \overline{v}}{2} \tag{9}$$

Last, if $(w_1, v_2) \in \mathcal{P}$, Candidate 1 wins if and only if both $v_1 \geq \overline{v} - w_1$ and $w_2 \leq w_1$, hence with probability

$$\Pr(A|(w_1, v_2) \in \mathcal{P} \text{ and C1 wins}) = \frac{w_1^2}{\overline{v}} \tag{10}$$

The conditional expectation of Candidate 1's quality in this case is

$$E(w_1 + v_1|(w_1, v_2) \in \mathcal{P} \text{ and C1 wins}) = w_1 + \overline{v} - \frac{w_1}{2} \tag{11}$$

Integrating over the respective parameter areas and summing up gives

$$\begin{aligned} E(v_1 + w_1|\text{A wins}) \Pr(\text{A wins}) &= \int_0^{\overline{v}-1} \left(\frac{\overline{v} - v_2}{\overline{v}}\right)^2 \left(\frac{1}{2} + \frac{\overline{v} + v_2}{2}\right) dv_2 \\ + \int_0^1 \int_{\overline{v}-1}^{\overline{v}-w_1} \left(\frac{\overline{v} - v_2}{\overline{v}}\right)^2 \left(w_1 + \frac{\overline{v} + v_2}{2}\right) dv_2 dw_1 &+ \int_0^1 \int_{\overline{v}-w_1}^{\overline{v}} \left(\frac{w_1}{\overline{v}}\right)^2 \left(\overline{v} + \frac{w_1}{2}\right) dv_2 dw_1 \\ = \frac{1}{4} + \frac{1}{3}\overline{v} + \frac{1}{60\overline{v}^2}. \end{aligned} \tag{12}$$

Dividing both sides by 1/2, the probability that A wins, gives the expected quality of the winning candidate in equilibrium

$$EU_1 = \frac{1}{2} + \frac{2}{3}\overline{v} + \frac{1}{30\overline{v}^2}, \tag{13}$$

if $\overline{v} \geq 1$. If $\overline{v} < 1$, we can use the fact that EU_1 is linearly homogeneous in $(\overline{v}, \overline{w})$ to calculate that

$$EU_1 = \overline{v} \left[\frac{1}{2} + \frac{2}{3}\frac{1}{\overline{v}} + \frac{1}{30\frac{1}{\overline{v}^2}}\right] = \frac{2}{3} + \frac{\overline{v}}{2} + \frac{\overline{v}^3}{30}. \tag{14}$$

Case 2. Consider now the case that candidates can only campaign positively, so that the electorate chooses the candidate with the better w and gets a random draw from the v distribution. Since

$$E(\max(w_1, w_2)) = 2\int_0^1 \int_{w_2}^1 w_1 dw_1 dw_2 = \frac{2}{3}$$

and $E(v) = \overline{v}2$, the expected quality of the winning candidate with only positive campaigning is

$$EU_2 = \frac{2}{3} + \frac{\overline{v}}{2}. \tag{15}$$

This implies that

$$EU_1 - EU_2 = \begin{cases} \frac{\overline{v}-1}{6} + \frac{1}{30\overline{v}^2} & \text{for } \overline{v} \geq 1 \\ \frac{\overline{v}^3}{30} & \text{for } \overline{v} \leq 1 \end{cases}. \tag{16}$$

Case 3. For the case of perfect information, suppose first that $\overline{v} \geq 1$. Then, the cumulative distribution function for $v + w$ is

$$G(x) = \begin{cases} \frac{x^2}{2\overline{v}} & \text{for } x \leq 1 \\ \frac{x-(1/2)}{\overline{v}} & \text{for } x \in (1, \overline{v}) \\ x - \frac{\overline{v}}{2} - \frac{(x-1)^2}{2\overline{v}} & \text{for } x \geq \overline{v} \end{cases} \tag{17}$$

The cumulative distribution function of $\max(v_1 + w_1, v_2 + w_2)$ is simply G^2. This allows us to calculate that, for $\overline{v} \geq 1$,

$$EU_3 = E(\max(v_1 + w_1, v_2 + w_2)) = \int_0^{\overline{v}+1} (1 - G^2(x))dx = \frac{1}{2} + \frac{2}{3}\overline{v} + \frac{5\overline{v} - 1}{60\overline{v}^2}. \tag{18}$$

Similarly, if $\overline{v} < 1$, then

$$EU_3 = E(\max(v_1 + w_1, v_2 + w_2)) = \overline{v} \left[\frac{1}{2} + \frac{2}{3}\frac{1}{\overline{v}} + \frac{5\frac{1}{\overline{v}} - 1}{60\frac{1}{\overline{v}^2}} \right] = \frac{2}{3} + \frac{\overline{v}}{2} + \frac{5\overline{v}^2 - \overline{v}^3}{60}. \tag{19}$$

Substituting the results so far in the definition of Θ yields

$$\Theta(\overline{v}) = \begin{cases} \frac{10\overline{v}^3 - 10\overline{v}^2 + 2}{10\overline{v}^3 + 5\overline{v} - 1} & \text{for } \overline{v} \geq 1 \\ \frac{2\overline{v}^3}{10 + 5\overline{v}^2 - \overline{v}^3} & \text{for } \overline{v} \leq 1 \end{cases}. \tag{20}$$

Obviously, $\Theta(\overline{v}) > 0$ for all $\overline{v}$. Taking the derivative of Θ gives

$$\Theta'(\overline{v}) = \begin{cases} \frac{10(10\overline{v}^4 + 10\overline{v}^3 - 14\overline{v}^2 + 2\overline{v} - 1)}{(10\overline{v}^3 + 5\overline{v} - 1)^2} & \text{for } \overline{v} \geq 1 \\ \frac{10\overline{v}^2(6 + \overline{v}^2)}{(10 + 5\overline{v}^2 - \overline{v}^3)^2} & \text{for } \overline{v} \leq 1 \end{cases}. \tag{21}$$

Hence, $\Theta' > 0$ for all $\overline{v}$. The remaining claims in the proposition are easy to verify by substitution in (20) (for $\overline{v} \to 0$) and using L'Hopital's rule (for $\overline{v} \to \infty$).

3 Welfare in the alternative signal system of Section 4.2

We derive here the expected quality of the winning candidate in the alternative signaling system of Section 4.2. Since exact calculations are burdensome, the values in Table 1 are calculated by numerical simulation.[1] For comparison purposes, we also provide the expected quality in equilibrium in the second column. In the last column, Ξ is the difference between the utilities in Case 1 and 4 relative to the utility difference between perfect and no information in the last column, so that

$$\Xi(v) := \frac{EU_4 - EU_1}{EU_3 - \frac{(1+\overline{v})}{2}}.$$

	Case 1 (equilibrium signaling)	Case 4 (alternative signaling)	$\Xi(\overline{v})$
$\bar{v} = 0.5$	0.921	0.924	0.016
$\bar{v} = 1$	1.200	1.219	0.082
$\bar{v} = 2$	1.842	1.847	0.013
$\bar{v} = 5$	3.835	3.837	0.002

Table 1: Expected quality $v + w$ of the winning candidate and $\Xi(\overline{v})$

The alternative signaling system improves the expected quality of the elected politician. For higher values of v, the effect of switching to the alternative system is smaller. This is quite intuitive, as information is almost always transmitted on v in both the alternative and the equilibrium signaling structures, and so the difference between these systems must be small. In particular, as v goes to infinity, expected utility in both systems converges to the full information utility.

Note that, while the alternative structure is more efficient in transmitting information, the probability of negative campaigning is exactly the same under the equilibrium signaling system and under the alternative system. In this sense, one cannot say that there is *too much* (or too little) negative campaigning in the equilibrium, just the parameter combinations in which negative campaigning is used is not optimal in the equilibrium signaling system.

[1] Specifically, for each value of $\overline{v}$, we have drawn 800,000 pairs of candidates and recorded the average quality of the candidate chosen under the alternative signaling system.