

Appendix

Throughout the appendix, V^{dec} refers to the profile of welfare under decentralization.

Lemma 1 proves Proposition 1. Lemma 2 gives conditions under which majority preferences on $\{L, R\}$ aggregate transitively.

Lemma 1 *For all (L, R) , the state equilibrium $x(L, R)$ characterized in Proposition 1 is unique and $x_n(L, R)$ is weakly increasing in θ, L, R and n . For any affine map $L(\cdot)$ and $R(\cdot)$, $x(L(\lambda), R(\lambda))$ is continuous, piecewise affine in λ and*

$$\frac{\partial \bar{x}(L, R)}{\partial L} = \frac{(1 + \beta)l}{N + \beta(l + r)} \text{ and } \frac{\partial \bar{x}(L, R)}{\partial R} = \frac{(1 + \beta)r}{N + \beta(l + r)}. \quad (5)$$

Proof: The first-order condition of the maximization program of each state gives that a state equilibrium is a fixed point of the best-response function:

$$f_n(x) = \max \left(L, \min \left(R, \frac{\theta_n + \beta \bar{x}}{1 + \beta} \right) \right), \quad \forall n. \quad (6)$$

Because f is a contraction for the sup norm on $[L, R]^N$, the state equilibrium exists and is unique. From Equation (6), $x_n(L, R)$ is weakly increasing in n . Because f is weakly increasing in x and in (θ, L, R) , Villas-Boas (1997, Theorem 4) implies that its fixed point $x(L, R)$ is weakly increasing in (θ, L, R) .

Observe that for all $A, B \subset N$, the set of $\{L, R\}$ such that the constraint $L \leq \frac{\theta_n + \beta \bar{x}}{1 + \beta}$ is binding for $n \in A$ and the constraint $\frac{\theta_n + \beta \bar{x}}{1 + \beta} \leq R$ is binding for $n \in B$ is a convex subset of \mathbb{R}^2 , because if x and x' are solutions of Equation (6) for (L, R) and (L', R') , respectively, then $\alpha x + (1 - \alpha)x'$ is solution for $\alpha(L, R) + (1 - \alpha)(L', R')$. The implicit function theorem implies that $x(L, R)$ is differentiable on the interior of these convex sets, and because there is a finite number of subsets A and B , for all affine maps $L(\cdot)$ and $R(\cdot)$, $x(L(\lambda), R(\lambda))$ is piecewise affine in λ . To obtain Equation (5), I differentiate Equation (6) w.r.t. L and R , sum over n , and solve for $\frac{\partial \bar{x}(L, R)}{\partial L}$ and $\frac{\partial \bar{x}(L, R)}{\partial R}$. ■

Lemma 2 *For all $L \leq L', R \leq R'$, $V_n(L', R') - V_n(L, R)$ is weakly increasing in n . In particular, majority preferences between $[L, R]$ and $[L', R']$ coincide with the preferences of the voters of the median state.*

Proof: Observe that the induced utility function $V_n(L, R)$ of state n can be written as

$$W_n(t_n, x) = \max_{y \in [L, R]} \left(-|y - t_n|^2 - \frac{\beta}{N} \sum_{m=1}^N |y - x_m|^2 \right), \quad (7)$$

for $x = x(L, R)$ and $t_n = \theta_n$. Let $y^*(t_n, x)$ be the maximizer of Equation (7). From the envelope theorem, for all $t_n \in \mathbb{R}$, $\frac{\partial W_n}{\partial t_n} = 2(y^*(t_n, x) - t_n)$, so

$$V_n(L, R) - V_m(L, R) = \int_{\theta_n}^{\theta_m} 2(y^*(t, x(L, R)) - t) dt,$$

which in turns implies

$$V_n(L', R') - V_n(L, R) - (V_m(L', R') - V_m(L, R)) = 2 \int_{\theta_n}^{\theta_m} \begin{pmatrix} y^*(t, x(L', R')) \\ -y^*(t, x(L, R)) \end{pmatrix} dt. \quad (8)$$

Observe that if $W(t_n, x, y)$ denotes the maximand of Equation (7), for all m , $\frac{\partial^2 W}{\partial y \partial x_m} > 0$. Hence W is supermodular in (y, x_m) and Topkis theorem implies that y^* is weakly increasing in x . From Lemma 1, if $L' \geq L$ and $R' \geq R$ then $x(L', R') \geq x(L, R)$, so the integrand in the right-hand side of Equation (8) is non-negative, which proves the first part of the lemma.

To prove the second part, notice that if the median voters strictly prefer (L', R') to (L, R) , so do the voters of the states $n \geq \mu$. ■

Lemma 3 *For all $L \leq R$, the state equilibrium $x(L, R)$ is equal to $x^{\text{dec}}(t)$ (see Equation (2)) where t is given by $t_n = \max(\underline{t}, \min(\bar{t}, \theta_n))$ for all n , with*

$$\begin{cases} \underline{t} = (1 + \beta)L - \beta \bar{x}(L, R), \\ \bar{t} = (1 + \beta)R - \beta \bar{x}(L, R). \end{cases} \quad (9)$$

At the decentralized equilibrium, for all $m \neq n$,

$$\frac{\partial V_n^{\text{dec}}}{\partial \theta_p} = \frac{2\beta}{N(1 + \beta)} (\theta_n - x_p^{\text{dec}}). \quad (10)$$

Proof: The map $\theta \rightarrow x^{\text{dec}}(\theta)$ defined in Equation (2) can be inverted as follows: $\theta_n = (1+\beta)x_n^{\text{dec}} - \beta x^{\text{dec}}$. To obtain the profile of type t in Equation (9), I substitute $x^{\text{dec}} = x(L, R)$ in the previous expression and use Equation (6).

Observe that for all n , the welfare under decentralization is given by $V_n^{\text{dec}} = W_n(\theta_n, x^{\text{dec}}(\theta))$ where W is defined in Equation (7) with $[L, R] = [\theta_1, \theta_N]$. The envelope theorem for $p \neq n$ gives

$$\begin{aligned} \frac{\partial V_n^{\text{dec}}}{\partial \theta_p} &= \frac{2\beta}{N} \sum_{m \neq p} (x_n^{\text{dec}} - x_m^{\text{dec}}) \frac{\partial x_m^{\text{dec}}}{\partial \theta_p} + \frac{2\beta}{N} (x_n^{\text{dec}} - x_p^{\text{dec}}) \frac{\partial x_p^{\text{dec}}}{\partial \theta_p}, \\ &= \frac{2\beta}{N(1+\beta)} \left[\sum_{m \neq p} (x_n^{\text{dec}} - x_m^{\text{dec}}) \frac{\beta}{N} + (x_n^{\text{dec}} - x_p^{\text{dec}}) \left(1 + \frac{\beta}{N}\right) \right], \quad (11) \end{aligned}$$

where $\frac{\partial x_m^{\text{dec}}}{\partial \theta_p}$ and $\frac{\partial x_p^{\text{dec}}}{\partial \theta_p}$ are derived from Equation (2). To obtain Equation (10), I substitute the first-order condition of Equation (7), i.e., $\frac{\beta}{N} \sum_m (x_n^{\text{dec}} - x_m^{\text{dec}}) = \theta_n - x_n^{\text{dec}}$, in Equation (11). ■

Proof of Proposition 2. *Step 1: the voters of the median state are pivotal.* $\Gamma - \Delta$ and $\Gamma + \Delta$ are increasing in Γ , so Lemma 2 implies that the majority preferences on Γ coincide with the preferences of the median state, so $\Gamma(\Delta)$ are the most preferred Γ of the voters of the median state.

Step 2: at the most preferred Γ of the voters of the median state, their policy is unconstrained. With the notation of Lemma 3, for all θ, Δ and for almost all Γ at which the median state is unconstrained,

$$\begin{aligned} \frac{\partial V_\mu}{\partial \Gamma}(\Gamma, \Delta) &= \frac{\partial [V_\mu^{\text{dec}}((\max(t, \min(\bar{t}, \theta_n)))_n)]}{\partial \Gamma} \\ &= \sum_{n > N-r(L, R)} \frac{\partial [V_\mu^{\text{dec}}(t)]}{\partial t_n} \frac{\partial \bar{t}}{\partial \Gamma} + \sum_{n \leq l(L, R)} \frac{\partial [V_\mu^{\text{dec}}(t)]}{\partial t_n} \frac{\partial t}{\partial \Gamma}. \quad (12) \end{aligned}$$

The substitution of Equation (5) in Equation (9) implies

$$\frac{\partial \bar{t}}{\partial \Gamma} = \frac{\partial t}{\partial \Gamma} = 1 + \beta - \beta \frac{(1+\beta)(l+r)}{N + \beta(l+r)} = \frac{(1+\beta)N}{N + \beta(l+r)}. \quad (13)$$

The substitution of Equations (10) and (13) in Equation (12) implies that for almost all Γ at which the median state is unconstrained,

$$\begin{aligned}\frac{\partial V_\mu}{\partial \Gamma}(\Gamma, \Delta) &= \frac{\beta}{N(1+\beta)} \left[l \times (\theta_\mu - \Gamma + \Delta) \frac{\partial t}{\partial \Gamma} + r \times (\theta_\mu - \Gamma - \Delta) \frac{\partial \bar{t}}{\partial \Gamma} \right] \\ &= \frac{\beta[l \times (\theta_\mu - \Gamma + \Delta) + r \times (\theta_\mu - \Gamma - \Delta)]}{N + \beta(l+r)}.\end{aligned}\quad (14)$$

From Equation (14), for almost all such Γ , $\frac{\partial V_\mu}{\partial \Gamma} > 0$ whenever $\Gamma + \Delta < \theta_\mu$ and $\frac{\partial V_\mu}{\partial \Gamma} < 0$ whenever $\Gamma - \Delta > \theta_\mu$. The same is true a fortiori when the median state is constrained by the federal bounds.¹ So the most preferred guideline Γ^* of the voters of the median state is such that $\theta_\mu \in [\Gamma^* - \Delta, \Gamma^* + \Delta]$. For all Γ , $\bar{x}(\Gamma, \Delta) \in [\Gamma - \Delta, \Gamma + \Delta]$, so necessarily, $\frac{\theta_\mu + \beta \bar{x}(\Gamma^*, \Delta)}{1+\beta} \in [\Gamma^* - \Delta, \Gamma^* + \Delta]$, so the median state is unconstrained.

Step 3: for all $\Gamma \in \Gamma(\Delta)$, $\Gamma = \theta_\mu + \frac{l-r}{l+r}\Delta$. Observe that $l(\Gamma, \Delta)$ and $r(\Gamma, \Delta)$ are weakly increasing and decreasing, respectively. So from Equation (14), at any point Γ^d of discontinuity of l or r , $\frac{\partial V_\mu}{\partial \Gamma}$ has an upward jump. Hence, the kinks of V_μ are all convex so V_μ is maximized at a differentiability point. To conclude, observe that from Equation (14), $\frac{\partial V_\mu}{\partial \Gamma} = 0$ implies $\Gamma = \theta_\mu + \frac{l-r}{l+r}\Delta$. ■

Proof of Proposition 3. *Step 1:* $\frac{\partial V_\mu^\theta}{\partial \Gamma}$ is weakly decreasing in θ_n for all $n \neq \mu$ and for almost all Γ such that at (Γ, Δ) , the median state is unconstrained. From Equation (14), for all $n \neq \mu$, $\frac{\partial V_\mu^\theta}{\partial \Gamma}$ depends on θ_n only through r or l . Therefore, $\frac{\partial V_\mu^\theta}{\partial \Gamma}$ is weakly decreasing in θ_n if the right-hand side of Equation (14) is decreasing in r and increasing in l . If $A(l, r)$ refers to the right-hand side of Equation (14),

$$A(l, r+1) - A(l, r) = \beta \frac{N(\theta_\mu - \Gamma - \Delta) - 2\beta l \Delta}{(N + \beta(l+r))(N + \beta(l+r+1))}. \quad (15)$$

For a fixed (Γ, Δ) and $(\theta_n)_{n \neq \mu}$, because l is decreasing in θ_μ , the numerator of Equation (15) is increasing in θ_μ . Let t_μ be the largest θ_μ at which state μ is unconstrained. Because the median state is unconstrained in equilibrium,

¹ For instance, if the constraint $x_\mu \leq \Gamma + \Delta$ is binding, then $\Gamma + \Delta < \theta_\mu$ so the right-hand side of Equation (14) is positive. Using the envelope theorem on Equation (7) for $x = x(\Gamma, \Delta)$, $\frac{\partial V_\mu}{\partial \Gamma}$ is given by Equation (14) plus the Lagrange multiplier of the constraint $x_\mu \leq \Gamma + \Delta$, which is positive. Therefore, $\frac{\partial V_\mu}{\partial \Gamma} > 0$.

from what precedes, to prove that $A(l, r)$ is weakly decreasing in r , it suffices to prove that the numerator of Equation (15) is negative at $\theta_\mu = t_\mu$. From Equation (6), at $\theta_\mu = t_\mu$, $x_\mu = \frac{\theta_\mu + \beta \bar{x}}{1 + \beta} = \Gamma + \Delta$, so

$$t_\mu - \Gamma - \Delta = \beta(\Gamma + \Delta - \bar{x}). \quad (16)$$

Moreover,

$$\bar{x} \leq \frac{(N - l) \times (\Gamma + \Delta) + l \times (\Gamma - \Delta)}{N}.$$

The substitution of the above inequality in Equation (16) implies $t_\mu - \Gamma - \Delta \leq 2\beta \frac{l\Delta}{N}$, which shows that the right-hand side of Equation (15) is negative. Similar algebra shows that $A(l, r)$ is decreasing in l .

Step 2: $\Gamma^\theta(\Delta)$ is weakly decreasing in θ_n in the strong set order sense for $n \neq \mu$. This follows directly from Step 1, Topkis Theorem and Proposition 2

In what follows, t is a profile of type which is skewed to the right and t^s is defined by $t_n^s = \frac{t_n + (2t_\mu - t_{2\mu - n})}{2}$ for all n .

Step 3: if there exists $G \in \Gamma^t(\Delta)$ such that $G > t_\mu$, then $2\theta_\mu - G \in \Gamma^t(\Delta)$. By construction, t^s is symmetric around t_μ (i.e., for all n , $t_n^s = 2t_\mu^s - t_{2\mu - n}^s$), $t_\mu^s = t_\mu$ and $t_n^s < t_n$ for $n \neq \mu$. By symmetry, $\Gamma^{t^s}(\Delta)$ is symmetric around t_μ so there exists $G' \in \Gamma^{t^s}(\Delta)$ such that $G' \leq t_\mu$. Because $G' < G$ and $t^s \leq t$, from Step 2, $G' \in \Gamma^t(\Delta)$ and $G \in \Gamma^{t^s}(\Delta)$. By symmetry, $2\theta_\mu - G \in \Gamma^{t^s}(\Delta)$ and because $2\theta_\mu - G \leq G$, Step 2 implies that $2\theta_\mu - G \in \Gamma^t(\Delta)$.

Step 4: if there exists $G \in \Gamma^t(\Delta)$ such that $G > t_\mu$, then for almost all Γ in $[2\theta_\mu - G, G]$, $l^{t^s}(\Gamma, \Delta) = l^t(\Gamma, \Delta)$ and $r^{t^s}(\Gamma, \Delta) = r^t(\Gamma, \Delta)$. From Equation (14), $\frac{\partial V_\mu^\theta}{\partial \Gamma}$ depends on $(\theta_n)_{n \neq \mu}$ only through l and r . As shown in Step 1, $\frac{\partial V_\mu^\theta}{\partial \Gamma}$ is increasing in l and decreasing in r . Because $t^s \leq t$,

$$l^{t^s}(\Gamma, \Delta) \geq l^t(\Gamma, \Delta) \text{ and } r^{t^s}(\Gamma, \Delta) \leq r^t(\Gamma, \Delta), \quad (17)$$

which implies that for almost all Γ , $\frac{\partial V_\mu^{t^s}}{\partial \Gamma}(\Gamma, \Delta) \leq \frac{\partial V_\mu^t}{\partial \Gamma}(\Gamma, \Delta)$, with a strict inequality when one of the inequalities in Equation (17) is strict. From Step 3, the median state is indifferent between $\Gamma = G$ and $\Gamma = 2\theta_\mu - G$ at $\theta = t$ and $\theta = t^s$, i.e. $\int_{2\theta_\mu - G}^G \frac{\partial V_\mu^\theta}{\partial \Gamma}(\Gamma, \Delta) d\Gamma = 0$ for $\theta \in \{t, t^s\}$. From what precedes, this implies that the inequalities in Equation (17) hold with equality for almost all Γ in $[2\theta_\mu - G, G]$.

Step 5: for all $G \in \Gamma^t(\Delta)$, $G \leq t_\mu$. Suppose $G > t_\mu$. For all $\theta \in \mathbb{R}^N$ and $n < \mu$, let $\Gamma_n^\theta(\Delta)$ be a value of Γ at which l jumps from $n - 1$ to n , i.e.,

$\frac{\theta_n + \beta \bar{x}(\Gamma_n, \Delta)}{1 + \beta} = \Gamma_n - \Delta$. Because \bar{x} and θ_n are greater at $\theta = t$ than at $\theta = t^s$, strictly so for θ_n , $\Gamma_n^{t^s}(\Delta) < \Gamma_n^t(\Delta)$. Likewise, one can show that the Γ at which r jumps from $n - 1$ to n is strictly greater at $\theta = t$ than at $\theta = t^s$. Together with Step 4, this implies that l and r are constant on $[2\theta_\mu - G, G]$. From Proposition 3, $G = 2\theta_\mu - G = \theta_\mu + \frac{l-r}{l+r}\Delta$ so $G = \theta_\mu$. ■

The next lemma characterizes $\mathbf{\Gamma}(\Delta)$ in the case of a triadic federation.

Lemma 4 *In a triadic federation (see Definition 1),*

$$\Delta > \min\{D_\lambda^o, D_\rho^o\} \Rightarrow \mathbf{\Gamma}(\Delta) = \begin{cases} \{\theta_\mu - \Delta\} & \text{if } D_\lambda^o < D_\rho^o \\ \{\theta_\mu + \Delta\} & \text{if } D_\lambda^o > D_\rho^o \\ \{\theta_\mu - \Delta, \theta_\mu + \Delta\} & \text{if } D_\lambda^o = D_\rho^o. \end{cases},$$

$$\Delta < \min\{D_\lambda^o, D_\rho^o\} \Rightarrow \mathbf{\Gamma}(\Delta) = \left\{ \theta_\mu + \frac{\lambda - \rho}{\lambda + \rho} \Delta \right\}.$$

For $\Delta \in]0, \min\{D_\lambda^o, D_\rho^o\}[$, $U_1(\theta, \mathbf{\Gamma}(\theta, \Delta), \Delta)$ and $U_N(\theta, \mathbf{\Gamma}(\theta, \Delta), \Delta)$ are single-peaked in Δ with a peak at D_λ and D_ρ , respectively.

Proof: From Proposition 2, for all Δ , there are three possible equilibria at the second stage: either $(l, r) = (0, \rho)$ and $\Gamma = \theta_\mu - \Delta$, or $(l, r) = (\lambda, 0)$ and $\Gamma = \theta_\mu + \Delta$, or $(l, r) = (\lambda, \rho)$ and $\Gamma = \theta_\mu + \frac{\lambda - \rho}{\lambda + \rho} \Delta$.

Step 1: derivation of the state equilibrium when $(l, r) = (0, \rho)$ and $\Gamma = \theta_\mu - \Delta$: In this case, $x(\theta_\mu - \Delta, \Delta)$ as characterized in Equation (6) is given by

$$\begin{cases} x_1 = \frac{\theta_1 + \frac{\beta}{N}(\lambda x_1 + (N - \lambda - \rho)x_2 + \rho x_N)}{1 + \beta} = \frac{(N + \beta\lambda + \beta\rho)\theta_1 + (\beta^2\rho + N\beta - \beta\lambda)\theta_\mu}{(\beta + 1)(N + \beta\rho)} \\ x_\mu = \frac{\theta_\mu + \frac{\beta}{N}(\lambda x_1 + (N - \lambda - \rho)x_2 + \rho x_N)}{1 + \beta} = \frac{(N + \beta^2\rho + N\beta - \beta\lambda + \beta\rho)\theta_\mu + \beta\lambda\theta_1}{(\beta + 1)(N + \beta\rho)} \\ x_N = \theta_\mu \end{cases}, \quad (18)$$

where the second equality is simply the solution of the system described by the first equality. This solution is possible at Δ iff the leftist states are indeed unconstrained at $(\Gamma = \theta_\mu - \Delta, \Delta)$, i.e.,

$$x_1 \geq \theta_\mu - 2\Delta \Leftrightarrow \Delta \geq D' \equiv \frac{N + \beta\lambda + \beta\rho}{2(\beta + 1)(N + \beta\rho)}(\theta_\mu - \theta_1).$$

In this case, the substitution of Equation (18) in Equation (1) implies

$$V_\mu(\theta_\mu - \Delta, \Delta) = -\frac{\beta\lambda(N + \beta\lambda + \beta\rho)}{N(\beta + 1)^2(N + \beta\rho)}(\theta_\mu - \theta_1)^2. \quad (19)$$

Step 2: derivation of the state equilibrium when $(l, r) = (\lambda, 0)$ and $\Gamma = \theta_\mu + \Delta$: In this case, $x(\theta, \theta_\mu + \Delta, \Delta)$ is given by

$$\begin{cases} x_1 = \theta_\mu \\ x_\mu = \frac{\theta_\mu + \frac{\beta}{N}(\lambda x_1 + (N - \lambda - \rho)x_2 + \rho x_N)}{1 + \beta} = \frac{(N + \beta^2 \lambda + N\beta - \beta\rho + \beta\lambda)\theta_\mu + \beta\rho\theta_N}{(\beta + 1)(N + \beta\lambda)} \\ x_N = \frac{\theta_N + \frac{\beta}{N}(\lambda x_1 + (N - \lambda - \rho)x_2 + \rho x_N)}{1 + \beta} = \frac{(N + \beta\rho + \beta\lambda)\theta_N + (\beta^2 \lambda + N\beta - \beta\rho)\theta_\mu}{(\beta + 1)(N + \beta\lambda)} \end{cases} \quad (20)$$

The substitution of Equation (20) in Equation (1) implies

$$V_\mu(\theta_\mu + \Delta, \Delta) = -\frac{\beta\rho(N + \beta\lambda + \beta\rho)}{N(\beta + 1)^2(N + \beta\lambda)}(\theta_N - \theta_\mu)^2. \quad (21)$$

From Proposition 2, for any Δ , $\Gamma(\Delta)$ are the most preferred Γ of the voters of the median state. From Equations (19) and (21), they strictly prefer $\Gamma = \theta_\mu - \Delta$ to $\Gamma = \theta_\mu + \Delta$ if and only if

$$\lambda(N + \beta\lambda)(\theta_\mu - \theta_1)^2 < \rho(N + \beta\rho)(\theta_N - \theta_\mu)^2 \Leftrightarrow D_\lambda^o < D_\rho^o. \quad (22)$$

Step 3: derivation of the state equilibrium when $(l, r) = (\lambda, \rho)$ and $\Gamma = \theta_\mu + \frac{\lambda - \rho}{\lambda + \rho}\Delta$: In this case, $x(\theta_\mu + \frac{\lambda - \rho}{\lambda + \rho}\Delta, \Delta)$ is given by

$$\begin{cases} x_1 = \theta_\mu - \frac{2\rho}{\lambda + \rho}\Delta, \\ x_\mu = \theta_\mu, \\ x_N = \theta_\mu + \frac{2\lambda}{\lambda + \rho}\Delta. \end{cases} \quad (23)$$

Notice that this solution is possible at Δ if and only if states 1 and N are indeed constrained at $(\Gamma, \Delta) = (\theta_\mu + \frac{\lambda - \rho}{\lambda + \rho}\Delta, \Delta)$, i.e.,

$$\begin{aligned} \frac{\theta_1 + \frac{\beta}{N}(\lambda x_1 + (N - \lambda - \rho)x_\mu + \rho x_N)}{1 + \beta} &\leq \Gamma - \Delta \quad \text{and} \\ \frac{\theta_N + \frac{\beta}{N}(\lambda x_1 + (N - \lambda - \rho)x_\mu + \rho x_N)}{1 + \beta} &\geq \Gamma + \Delta, \end{aligned}$$

which is equivalent to

$$\Delta \leq D'' \equiv \frac{\lambda + \rho}{2(1 + \beta)} \min \left\{ \frac{\theta_\mu - \theta_1}{\rho}, \frac{\theta_N - \theta_\mu}{\lambda} \right\}.$$

The substitution of Equation (23) in Equation (1) implies that at $(\Gamma, \Delta) = (\theta_\mu + \frac{\lambda-\rho}{\lambda+\rho}\Delta, \Delta)$,

$$\begin{cases} V_1 = -\left(\theta_1 - \theta_\mu + \frac{2\rho}{\lambda+\rho}\Delta\right)^2 - \frac{\beta}{N} \left((N - \lambda - \rho)\left(\frac{2\rho}{\lambda+\rho}\right)^2 + 4\rho\right) \Delta^2, \\ V_\mu = -4\frac{\beta}{N} \frac{\lambda\rho}{\lambda+\rho} \Delta^2, \\ V_N = -\left(\theta_N - \theta_\mu - \frac{2\lambda}{\lambda+\rho}\Delta\right)^2 - \frac{\beta}{N} \left((N - \lambda - \rho)\left(\frac{2\rho}{\lambda+\rho}\right)^2 + 4\lambda\right) \Delta^2. \end{cases} \quad (24)$$

Step 4: derivation of the second stage equilibrium: from Equations (19) and (24), the voters of the median state prefer $\Gamma = \theta_\mu + \frac{\lambda-\rho}{\lambda+\rho}\Delta$ to $\Gamma = \theta_\mu - \Delta$ if and only if

$$\Delta \leq \sqrt{\frac{(\lambda + \rho)(N + \beta\lambda + \beta\rho)}{4\rho(\beta + 1)^2(N + \beta\rho)}}(\theta_\mu - \theta_1) = D_\lambda^o.$$

A symmetric reasoning shows that the voters of the median state prefer $\Gamma = \theta_\mu + \frac{\lambda-\rho}{\lambda+\rho}\Delta$ to $\Gamma = \theta_\mu - \Delta$ if and only if $\Delta \leq D_\rho^o$. Together with Equation (22), this proves that if $D_\lambda^o \leq D_\rho^o$, for $\Delta > D_\lambda^o$, $\Gamma(\Delta) = \{\theta_\mu - \Delta\}$ and for $\Delta < D_\rho^o$, $\Gamma(\Delta) = \{\theta_\mu + \frac{\lambda-\rho}{\lambda+\rho}\Delta\}$. Likewise, if $D_\lambda^o \geq D_\rho^o$, for $\Delta > D_\rho^o$, $\Gamma(\Delta) = \{\theta_\mu + \Delta\}$ and for $\Delta < D_\lambda^o$, $\Gamma(\Delta) = \{\theta_\mu + \frac{\lambda-\rho}{\lambda+\rho}\Delta\}$. Simple algebra shows that the feasibility constraints, i.e. $D'' \geq \min\{D_\lambda^o, D_\rho^o\} \geq D'$, are always satisfied,² which completes the proof of the first part of the lemma.

Step 5: derivation of the induced preferences of state 1 and N on Δ : if $\Delta < \min\{D_\lambda^o, D_\rho^o\}$, from Equation (24), $\frac{\partial U_1(\theta_\mu + \frac{\lambda-\rho}{\lambda+\rho}\Delta, \Delta)}{\partial \Delta} = 0$ iff

$$-\frac{4\rho}{\lambda + \rho} \left(\theta_1 - \theta_\mu + \frac{2\rho}{\lambda + \rho}\Delta\right) - \frac{2\beta}{N} \left((N - \lambda - \rho)\left(\frac{2\rho}{\lambda + \rho}\right)^2 + 4\rho\right) \Delta = 0,$$

² Suppose to fix ideas that $D_\lambda^o \leq D_\rho^o$. If $\frac{|\theta_\mu - \theta_1|}{\rho} \leq \frac{|\theta_N - \theta_\mu|}{\lambda}$, $D'' \geq D_\lambda^o \geq D'$ can be rewritten as

$$\sqrt{\frac{(\lambda + \rho)(N + \beta\rho)}{(N + \beta(\lambda + \rho))\rho}} \geq 1 \geq \sqrt{\frac{(N + \beta(\lambda + \rho))\rho}{(\lambda + \rho)(N + \beta\rho)}},$$

which is satisfied since $\frac{N + \beta x}{x}$ is decreasing in x and $\lambda + \rho > \rho$. If $\frac{|\theta_\mu - \theta_1|}{\rho} > \frac{|\theta_N - \theta_\mu|}{\lambda}$, $D'' \geq D_\lambda^o \geq D'$ holds when

$$\frac{|\theta_N - \theta_\mu|}{|\theta_\mu - \theta_1|} \geq \sqrt{\frac{(N + \beta(\lambda + \rho))\lambda^2(N + \beta\lambda)}{(\lambda + \rho)(N + \beta\lambda)\rho(N + \beta\rho)}},$$

which is true since $D_\lambda^o \leq D_\rho^o$ implies $\frac{|\theta_N - \theta_\mu|}{|\theta_\mu - \theta_1|} \geq \sqrt{\frac{\lambda(N + \beta\lambda)}{\rho(N + \beta\rho)}}$, and from above,

$$\sqrt{\frac{(N + \beta(\lambda + \rho))\lambda}{(\lambda + \rho)(N + \beta\lambda)}} < 1.$$

the solution of which is $\Delta = D_\lambda$. Likewise, $\frac{\partial V_N(\theta_\mu + \frac{\lambda-\rho}{\lambda+\rho}\Delta, \Delta)}{\partial \Delta} = 0$ iff $\Delta = D_\rho$. ■

Proof of Proposition 4. Lemma 4 shows that $|\mathbf{\Gamma}(\Delta) - \theta_\mu|$ is single-valued and increasing in Δ on $[0, \min\{D_\lambda^o, D_\rho^o\}[$ and on $] \min\{D_\lambda^o, D_\rho^o\}, +\infty[$. To complete the proof, it suffices to notice that $\theta_\mu + \frac{\lambda-\rho}{\lambda+\rho}\Delta$ is closer to θ_μ than $\theta_\mu \pm \Delta$. ■

Lemmas 5 and 6 show that $\mathbf{\Gamma}(\Delta)$ has a finite number of discontinuity points and characterize its behavior at the continuity and discontinuity points. Lemmas 7 and 8 derive some necessary and sufficient conditions of the local federal equilibrium.

Lemma 5 *With the following notations*

$$\begin{cases} \mathbf{L}(\Delta) = \{l(\Gamma, \Delta) : \Gamma \in \mathbf{\Gamma}(\Delta)\}, \\ \mathbf{R}(\Delta) = \{r(\Gamma, \Delta) : \Gamma \in \mathbf{\Gamma}(\Delta)\}, \end{cases}$$

there exists D_1, \dots, D_I (with the convention that $D_0 = 0, D_{I+1} = +\infty$) such that for all $i = 0, \dots, I$, $(\mathbf{L}, \mathbf{R})(\Delta)$ is constant on $]D^i, D^{i+1}[$ and for all $i = 1, \dots, I$,

$$\left\{ \lim_{\Delta \nearrow D^i} (\mathbf{L}, \mathbf{R})(\Delta), \lim_{\Delta \searrow D^i} (\mathbf{L}, \mathbf{R})(\Delta) \right\} \subset (\mathbf{L}, \mathbf{R})(D^i). \quad (25)$$

Proof: From Proposition 2, $(\mathbf{L}, \mathbf{R})(\Delta) = \arg \max_{(l,r)} V_\mu(\theta_\mu + \frac{l-r}{l+r}\Delta, \Delta)$. From Lemma 1, for all (l, r) , $V_\mu(\theta_\mu + \frac{l-r}{l+r}\Delta, \Delta)$ is piecewise quadratic in Δ . Because (l, r) can only take a finite number of values, $(\mathbf{L}, \mathbf{R})(\Delta)$ must be piecewise constant. The Berge maximum theorem implies that $(\mathbf{L}, \mathbf{R})(\Delta)$ is upper hemi-continuous in Δ , which implies Equation (25). ■

Lemma 6 *The induced preferences of the voters of the median state on $x(\mathbf{\Gamma}(\Delta), \Delta)$ are decreasing in Δ . With the notations of Lemma 5, for almost all type profile θ , $\mathbf{\Gamma}(\Delta)$ is single-valued except at its finitely many discontinuity points D_1, \dots, D_I . Whenever $\mathbf{\Gamma}(\Delta)$ is single-valued on some $]D^i, D^{i+1}[$, the unconstrained states $\{l+1, \dots, N-r\}$ have decreasing preferences over Δ on $]D^i, D^{i+1}[$ and*

$$\begin{cases} x_n(\mathbf{\Gamma}(\Delta), \Delta) = \theta_\mu - \frac{2r}{l+r}\Delta, & \text{for } n \leq l, \\ x_n(\mathbf{\Gamma}(\Delta), \Delta) = \theta_\mu + \frac{2l}{l+r}\Delta, & \text{for } n > N-r, \\ \frac{\partial [x_n(\mathbf{\Gamma}(\Delta), \Delta)]}{\partial \Delta} = 0, & \text{for } l < n \leq N-r. \end{cases} \quad (26)$$

Proof: Suppose that $\Gamma(\Delta)$ is not single-valued on $]D^i, D^{i+1}[$, then one can slightly perturb the type of a state which is constrained in one second stage equilibrium but not at the other so as to make the median state not indifferent anymore between the two guidelines. This shows that for almost all θ , $\Gamma(\Delta)$ is single-valued except at its discontinuity points.

Suppose $\Gamma(\Delta)$ is single-valued on $]D^i, D^{i+1}[$. Equation (26) for $n \leq l$ and $n > N - r$ is a corollary of Proposition 2. For $n \in \{l + 1, \dots, N - r\}$, from Proposition 1,

$$x_n(\Gamma(\Delta), \Delta) = \frac{\theta_n + \beta \bar{x}(\Gamma(\Delta), \Delta)}{1 + \beta}.$$

I differentiate with respect to Δ and substitute Equation (5) to get

$$\frac{\partial[x_n(\Gamma(\Delta), \Delta)]}{\partial \Delta} = \frac{\beta}{1 + \beta} \frac{(1 + \beta)(l + r) \frac{l-r}{l+r} + (1 + \beta)(r - l)}{N + \beta(l + r)} = 0.$$

The substitution of Equation (26) in Equation (1) shows that unconstrained states prefer less discretion on $]D^i, D^{i+1}[$.

From Proposition 2, the median voters are indifferent between all elements of $\Gamma(\Delta)$ and their policy is unconstrained at (Γ, Δ) for any $\Gamma \in \Gamma(\Delta)$. From what precedes, this implies that the median voters always prefer less discretion. ■

Lemma 7 *If (Γ^e, Δ^e) is a local federal equilibrium, then $\Gamma(\Delta)$ is single valued on $]\Delta^e - \varepsilon, \Delta^e[$ for some positive ε and $\Gamma^e = \lim_{\Delta \nearrow \Delta^e} \Gamma(\Delta)$.*

Proof: Suppose that $\Gamma(\Delta)$ is not single-valued on $]\Delta^e - \varepsilon, \Delta^e[$. From Equation (25), $\Gamma(\Delta^e)$ is not single-valued. Let $\Gamma^o \in \Gamma(\Delta^e)$ such that $\Gamma^o \neq \Gamma^e$, say $\Gamma^o < \Gamma^e$ for concreteness. From Proposition 2, $\Gamma^o = \theta_\mu + \frac{l^o - r^o}{l^o + r^o} \Delta^e$ where $(l^o, r^o) = (l, r)(\Gamma^o, \Delta^e)$ and the voters from the median state are indifferent between (Γ^o, Δ^e) and (Γ^e, Δ^e) . From Lemma 2, the voters from leftist states strictly prefer (Γ^o, Δ^e) to (Γ^e, Δ^e) . By continuity, they strictly prefer $(\theta_\mu + \frac{l^o - r^o}{l^o + r^o}(\Delta^e - \varepsilon'), \Delta^e - \varepsilon')$ for some small $\varepsilon' > 0$. From Lemma 6, so do the median voters. From Lemma 5, $\theta_\mu + \frac{l^o - r^o}{l^o + r^o}(\Delta^e - \varepsilon') \in \Gamma(\Delta^e - \varepsilon')$, hence $(\theta_\mu + \frac{l^o - r^o}{l^o + r^o}(\Delta^e - \varepsilon'), \Delta^e - \varepsilon')$ is a valid deviation from (Γ^e, Δ^e) and (Γ^e, Δ^e) cannot be a local federal equilibrium.

If $\Gamma^e \neq \lim_{\Delta \nearrow \Delta^e} \Gamma(\Delta)$, the same reasoning shows that $(\theta_\mu + \frac{l^o - r^o}{l^o + r^o}(\Delta^e - \varepsilon), \Delta^e - \varepsilon)$ is majority preferred to (Γ^e, Δ^e) where $(l^o, r^o) = (l, r)(\Gamma^o, \Delta^e)$ and $\Gamma^o = \lim_{\Delta \nearrow \Delta^e} \Gamma(\Delta)$. ■

Lemma 8 *With the notations of Lemma 5, (Γ^e, Δ^e) is a local federal equilibrium if and only if there exists $i > 0$ such that one of the following is true:*

- (a) $\Delta^e \in]D^{i-1}, D^i[$, $\Gamma(\Delta)$ is single-valued on $]D^{i-1}, D^i[$, $\Gamma(\Delta^e) = \{\Gamma^e\}$ and majority preferences on Δ are strictly single peaked on $]D^{i-1}, D^i[$ with a peak at Δ^e .
- (b) $\Delta^e = D_i$, $\Gamma(\Delta)$ is single-valued on $]D^{i-1}, D^i[$, $\Gamma^e = \lim_{\Delta \nearrow D_i} \Gamma(\Delta)$ and majority preferences on Δ are increasing on $]D^{i-1}, D^i[$.

Proof: If (a) is satisfied, (Γ^e, Δ^e) is clearly a local federal equilibrium. Reciprocally, suppose that (Γ^e, Δ^e) is a local federal equilibrium and $\Delta^e \in]D^{i-1}, D^i[$. From Lemmas 7 and 5, $\Gamma(\Delta)$ is single-valued on $]D^{i-1}, D^i[$. Therefore, the induced preferences of all voters on $x(\Gamma(\Delta), \Delta)$ are well-defined, quadratic and concave, and not flat by definition of a local federal equilibrium. Therefore, majority preferences are strictly quasi-concave on $]D^{i-1}, D^i[$, and the conclusion follows from the median voter theorem.

If (Γ^e, Δ^e) is a local federal equilibrium and $\Delta^e = D^i$, from Lemma 7 and $\Gamma(\Delta)$ is single-valued on $]D^{i-1}, D^i[$. Suppose majority preferences are not increasing on $]D^{i-1}, D^i[$. As argued earlier, majority preferences are quasi-concave on $]D^{i-1}, D^i[$, so they must be decreasing on $]D^i - \varepsilon, D^i[$ for some $\varepsilon > 0$. From Lemma 7, $\Gamma^e = \lim_{\Delta \nearrow D^i} \Gamma(\Delta)$ and from what precedes, (Γ^e, Δ^e) cannot be a local federal equilibrium.

Reciprocally, suppose (b) is satisfied, then by assumption, there exists $\varepsilon > 0$ such that for all $\Delta \in]\Delta^e - \varepsilon, \Delta^e[$, (Γ^e, Δ^e) is majority preferred to $(\Gamma(\Delta), \Delta)$. For all $\Gamma \in \Gamma(\Delta^e)$, the median voter is indifferent between (Γ^e, Δ^e) and (Γ, Δ^e) so from Lemma 2, (Γ^e, Δ^e) is majority preferred to (Γ, Δ^e) . Finally, let $\Delta^k \searrow \Delta^e$ and $\Gamma^k \in \Gamma(\Delta^k)$. One can restrict attention to sequences such that $(l, r)(\Gamma^k, \Delta^k) = (l^o, r^o)$ for all k and some (l^o, r^o) . Because $\Delta^k > \Delta^e$, Lemma 6 implies that the median state strictly prefer (Γ^e, Δ^e) to (Γ^k, Δ^k) . From Equation (3), $\Gamma^o \equiv \lim \Gamma^k$ exists. Suppose to fix ideas that $\Gamma^e < \Gamma^o$.³ For k sufficiently large, $\Gamma^e \pm \Delta^e < \Gamma^k \pm \Delta^k$ so from Lemma 2 and what precedes, (Γ^e, Δ^e) is strictly preferred to (Γ^k, Δ^k) by leftist states. Finally (Γ^e, Δ^e) is majority preferred to (Γ^k, Δ^k) for k sufficiently large. ■

³ As can be seen from the proof of Lemma 5, since $\Gamma(\theta, \Delta)$ is single-valued on $] \Gamma^e - \varepsilon, \Gamma^e[$, $\Gamma^e \neq \Gamma^o$.

Proof of Proposition 5. As $\Delta \rightarrow 0$, from Proposition 2, all states n such that $\theta_n \neq \theta_\mu$ must be constrained and $\Gamma(\Delta)$ is single valued. In this case, one can easily see from Equation (1) that $V_n(\Gamma(\Delta), \Delta)$ must be increasing in a neighborhood of $\Delta = 0$.⁴ For a majority of states, $\theta_n \equiv \theta_\mu$, so $\Delta = 0$ cannot be a local federal equilibrium. With the notations of Lemma 5, because $\Gamma(\Delta)$ is single valued on $]D^0, D^1[$, majority preferences are quasi-concave on $]D^0, D^1[$. If they are increasing on $]D^0, D^1[$, Lemma 8 implies that D^1 is a local federal equilibrium. If they are not increasing, from what precedes, they must be single-peaked and Lemma 8 implies that there exists a local federal equilibrium in $]D^0, D^1[$.

From Lemma 8, at any local federal equilibrium (Γ^e, Δ^e) , a majority of voters have preferences which are weakly increasing in Δ on $]\Delta^e - \varepsilon, \Delta^e[$. From Lemma 6, these voters are from states which are constrained by the federal bounds, which proves that a majority of states must be constrained at (Γ^e, Δ^e) . ■

Proof of Proposition 6. For the sake of brevity, I assume throughout the proof that $D_\lambda^o > D_\rho^o$. The cases $D_\lambda^o < D_\rho^o$ and $D_\lambda^o = D_\rho^o$ can be derived identically. On $]D_\rho^o, +\infty[$, from Lemma 4, the state equilibrium is constant (see Equation (18)) and equal to $x(\theta_\mu + D_\rho^o, D_\rho^o)$. Because $\lim_{\Delta \nearrow D_\rho^o} \Gamma(\Delta) \neq \theta_\mu + D_\rho^o$, from Lemma 7, $(\Gamma, \Delta) = (\theta_\mu + D_\rho^o, D_\rho^o)$ is not a local federal equilibrium. From Lemma 7, the only local federal equilibrium candidates are therefore $\left\{ (\theta_\mu + \frac{\lambda - \rho}{\lambda + \rho} \Delta, \Delta) : \Delta \leq D_\rho^o \right\}$. On $[0, D_\rho^o[$, the preferences of all voters on $x(\theta_\mu + \frac{\lambda - \rho}{\lambda + \rho} \Delta, \Delta)$ are quadratic and concave in Δ . Therefore, majority preferences are quasi-concave on $[0, D_\rho^o[$. Moreover, they are either single-peaked or strictly increasing on $[0, D_\rho^o[$.⁵

Case 1: majority preferences are increasing on $[0, D_\rho^o[$. From Lemma 7 $(\Gamma^e, \Delta^e) \equiv (\theta_\mu + \frac{\lambda - \rho}{\lambda + \rho} D_\rho^o, D_\rho^o)$ is the only local federal equilibrium. By assumption, (Γ^e, Δ^e) is majority preferred to $(\theta_\mu + \frac{\lambda - \rho}{\lambda + \rho} \Delta, \Delta)$ for $\Delta < D_\rho^o$. From Proposition 2, the voters of the median state are indifferent between (Γ^e, Δ^e) and $(\theta_\mu + D_\rho^o, D_\rho^o)$, from Lemma 2 the rightist and leftist voters have opposite preferences between these two alternatives, so (Γ^e, Δ^e) is majority

⁴ The first-order effect on $-|x_n - \theta_n|^2$ is positive while the first-order effect on $-\frac{\beta}{N} \sum_{m \neq n} |x_n - x_m|^2$ is zero.

⁵ They cannot be decreasing to the right of $\Delta = 0$ from Equation 24.

preferred to $(\theta_\mu + D_\rho^o, D_\rho^o)$. This shows that (Γ^e, Δ^e) is a Condorcet winner among all (Γ, Δ) such that $\Gamma \in \mathbf{\Gamma}(\Delta)$.

Case 2: majority preferences are single-peaked on $[0, D_\rho^o]$. From Lemma 4, $\Delta^e = \min\{D_\lambda, D_\rho\}$ and $\Delta^e \leq D_\rho^o$. From Lemma 7, $(\Gamma^e = \theta_\mu + \frac{\lambda-\rho}{\lambda+\rho}\Delta^e, \Delta^e)$ is the unique local federal equilibrium. By assumption, (Γ^e, Δ^e) is majority preferred to $(\theta_\mu + \frac{\lambda-\rho}{\lambda+\rho}\Delta, \Delta)$ for $\Delta \leq D_\rho^o$. From Lemma 6, the voters of the median state strictly prefer (Γ^e, Δ^e) to $(\theta_\mu + D_\rho^o, D_\rho^o)$. Moreover, because $\Delta^e \leq D_\rho^o$,

$$\theta_\mu + \frac{\lambda-\rho}{\lambda+\rho}\Delta^e \pm \Delta^e < \theta_\mu + D_\rho^o \pm D_\rho^o,$$

so from Lemma 2 (Γ^e, Δ^e) is majority preferred to $(D_\rho^o, \theta_\mu + D_\rho^o)$. Hence, (Γ^e, Δ^e) is a Condorcet winner among all (Γ, Δ) such that $\Gamma \in \mathbf{\Gamma}(\Delta)$.

To prove the last point, it suffices to notice that D_λ^o , D_ρ^o , D_λ , and D_ρ are all decreasing in β and increasing in $|\theta_\mu - \theta_1|$ and in $|\theta_\mu - \theta_N|$. ■

Proof of Proposition 7. The substitution of Equation (2) in Equation (1) yields

$$\begin{cases} V_n^{\text{dec}} = -\left(\frac{\beta}{1+\beta}(\theta_n - \bar{\theta})\right)^2 - \frac{\beta}{N} \sum_m \left(\frac{1}{1+\beta}(\theta_n - \theta_m)\right)^2, \\ W^{\text{dec}} = -N \frac{2\beta+\beta^2}{1+2\beta+\beta^2} \text{var}(\theta). \end{cases} \quad (27)$$

Let $(\theta^k)_{k \in \mathbb{N}}$ be such that $\frac{|\theta_N^k - \theta_\mu^k|}{|\theta_1^k - \theta_\mu^k|} \rightarrow \infty$. Without loss of generality, I can renormalize the type space and the policy space for each $k \in \mathbb{N}$ so that θ_1^k and θ_μ^k are constants, and thus for all $n \leq \mu$, θ_n^k is bounded, but $\theta_N^k \rightarrow +\infty$. Along this sequence $(\theta^k)_{k \in \mathbb{N}}$, the voters of the median state can always choose Γ to make sure that $x^{\theta^k}(\Gamma, \Delta)$ is bounded over k and Δ (e.g., $\Gamma = \theta_\mu - \Delta$). From Proposition 2, the voters of the median state are always pivotal on Γ , so $x^{\theta^k}(\mathbf{\Gamma}^{\theta^k}(\Delta), \Delta)$ must be bounded over k and Δ , otherwise $\liminf_k V_\mu^{\theta^k}(\mathbf{\Gamma}^{\theta^k}(\Delta), \Delta) = -\infty$. This implies that for all $n \in N$, $V_n^{\theta^k}(\Gamma, \Delta) = -(\theta_n^k)^2 + O(1)$, so $W^{\theta^k}(\Gamma, \Delta) \leq -N \text{var}(\theta^k) + O(1)$, and from (27), $W^{\theta^k}(\Gamma, \Delta) < W^{\theta^k, \text{dec}}$ for k sufficiently large. ■

Lemma 9 As $\beta \rightarrow 0$, for any selection of local federal equilibrium $(\Gamma^\beta, \Delta^\beta)$, for all constrained states n , $|x_n^\beta(\Gamma^\beta, \Delta^\beta) - \theta_n|$ is bounded away from 0.

Proof: To show that for all constrained states n , $|x_n^\beta(\Gamma^\beta, \Delta^\beta) - \theta_n|$ is bounded away from 0 as $\beta \rightarrow 0$, I proceed by contradiction: in what follows, β^k is a sequence such that $\beta^k \rightarrow 0$ and (Γ^k, Δ^k) is a sequence of local federal equilibrium for the profile of preferences (θ, β^k) such that $x_m^{\beta^k}(\Gamma^k, \Delta^k) \rightarrow \theta_m$ for some state m (independent of k) which is constrained for all k , and (\mathbf{L}, \mathbf{R}) refer to the map defined before Lemma 5. I assume w.l.o.g. that for all k , m is constrained by the right bound.

Step 1: there exists a subsequence of β^k such that $(\Gamma^k, \Delta^k) \rightarrow (\Gamma^o, \Delta^o)$, for all k , $l^{\beta^k}(\Gamma^k, \Delta^k) = l^o$ and $r^{\beta^k}(\Gamma^k, \Delta^k) = r^o$ for some Γ^o, Δ^o, l^o and r^o , and $\theta_m = \Gamma^o + \Delta^o$. Because (\mathbf{L}, \mathbf{R}) can only take a finite number of values and (Γ^k, Δ^k) is bounded, the limit exists. To see the last point, observe that for all k , $\bar{x}^{\beta^k}(\Gamma^k, \Delta^k) \leq \Gamma^k + \Delta^k$. So Proposition 1 implies that $\theta_m \geq x_m^{\beta^k}(\Gamma^k, \Delta^k)$, so $\theta_m \geq \Gamma^o + \Delta^o$. Because m is constrained by the right bound, Proposition 1 implies further that $\theta_m > \Gamma^k + \Delta^k$, so $\theta_m \geq \Gamma^o + \Delta^o$.

Step 2: there exists a subsequence of β^k such that $l^{\beta^k}(\Gamma, \Delta^k)$ and $r^{\beta^k}(\Gamma, \Delta^k)$ converge pointwise to two step functions $l(\Gamma)$ and $r(\Gamma)$ such that $\lim_{\Gamma \searrow \Gamma^o} r(\Gamma) \leq r^o - 1$ and $\lim_{\Gamma \searrow \Gamma^o} l(\Gamma) \geq l^o$. Observe that for a given k , $l^{\beta^k}(\Gamma, \Delta^k)$ and $r^{\beta^k}(\Gamma, \Delta^k)$ are functions of Γ which are piecewise constant, bounded and have a bounded number of points of discontinuities as $k \rightarrow \infty$. The existence of $l(\Gamma)$ and $r(\Gamma)$ follows from Bolzano–Weierstrass. From Step 1, for all $\Gamma > \Gamma^o$, for k sufficiently large, $\Gamma^k < \Gamma$ so $\lim_{k \rightarrow \infty} l^{\beta^k}(\Gamma, \Delta^k) \geq l^o$. From Step 1 again, $\theta_m = \Gamma^o + \Delta^o$ so for k sufficiently large, $r^{\beta^k}(\Gamma, \Delta^k) \leq N - m$. Because m is constrained for all k , $N - m + 1 \leq r^o$.

Step 3: there exists $\epsilon > 0$ and $c > 0$ such that

$$V_\mu^{\beta^k}(\Gamma^k - \epsilon, \Delta^k) - V_\mu^{\beta^k}(\Gamma^k, \Delta^k) = c\beta^k + o(\beta^k) \quad (28)$$

From Equation (14), for all $\epsilon > 0$,

$$\begin{aligned} & V_\mu^{\beta^k}(\Gamma^k + \epsilon, \Delta^k) - V_\mu^{\beta^k}(\Gamma^k, \Delta^k) \\ &= \frac{\beta^k}{N} \int_{\Gamma^k}^{\Gamma^k + \epsilon} \left(l^{\beta^k}(\Gamma, \Delta^k)(\theta_\mu - \Gamma + \Delta^k) \right. \\ & \quad \left. + r^{\beta^k}(\Gamma, \Delta^k)(\theta_\mu - \Gamma - \Delta^k) \right) d\Gamma + o(\beta^k). \end{aligned} \quad (29)$$

From Proposition 2, $\Gamma^k \rightarrow \theta_\mu + \frac{l^o - r^o}{l^o + r^o} \Delta^o$. Together with Step 2 and the dominated convergence theorem, this implies that

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \int_{\Gamma^k}^{\Gamma^k + \epsilon} [l^{\beta^k}(\Gamma, \Delta^k)(\theta_\mu - \Gamma + \Delta^k) + r^{\beta^k}(\Gamma, \Delta^k)(\theta_\mu - \Gamma - \Delta^k)] d\Gamma \\
&= \int_0^\epsilon \left[l \left(\theta_\mu + \frac{l^o - r^o}{l^o + r^o} \Delta^o + \varepsilon \right) \times \left(\frac{2r^o}{l^o + r^o} \Delta^o - \varepsilon \right) \right. \\
&\quad \left. + r \left(\theta_\mu + \frac{l^o - r^o}{l^o + r^o} \Delta^o + \varepsilon \right) \times \left(-\frac{2l^o}{l^o + r^o} \Delta^o - \varepsilon \right) \right] d\varepsilon \\
&\geq \int_0^\epsilon \left(-(l^o + r^o + 1)\varepsilon + \frac{2l^o}{l^o + r^o} \Delta^o \right) d\varepsilon. \tag{30}
\end{aligned}$$

Simple calculus shows that Equation (30) is positive for ϵ sufficiently small, which, together with Equation (29), completes the proof of Step 3.

To conclude, observe that Step 3 implies that $V_\mu^{\beta^k}(\Gamma^k - \epsilon, \Delta^k) > V_\mu^{\beta^k}(\Gamma^k, \Delta^k)$ for some k and some $\epsilon > 0$, which contradicts Proposition 2. ■

Proof of Proposition 8. Lemma 9 implies that as $\beta \rightarrow 0$, the left bound of the discretionary interval $\Gamma^\beta - \Delta^\beta$ is bounded above $\theta_1 + \varepsilon$ and that the right bound $\Gamma^\beta + \Delta^\beta$ is bounded below by $\theta_N - \varepsilon$ for some $\varepsilon > 0$ independent of β ; it should be clear from Equation (1) that as $\beta \rightarrow 0$, the optimal interval of discretion tends to $[\theta_1, \theta_N]$, which proves part (a).

Lemma 9 also implies that for all constrained states n , for any selection of local federal equilibrium $(\Gamma^\beta, \Delta^\beta)_{\beta > 0}$, $V_n^\beta(\Gamma^\beta, \Delta^\beta)$ is bounded away from 0 as $\beta \rightarrow 0$, and for all unconstrained states n , $V_n^\beta(\Gamma^\beta, \Delta^\beta) \rightarrow 0$. From Equation (27), for all $n \in N$, $V_n^{\text{dec}, \beta} \rightarrow 0$, which proves part (b). ■

Proof of Proposition 9. From Equation (27), as $\beta \rightarrow \infty$,

$$\lim_{\beta \rightarrow \infty} V_n^{\text{dec}, \beta} = -(\theta_n - \bar{\theta})^2. \tag{31}$$

Let $(\Gamma^\beta, \Delta^\beta)_{\beta > 0}$ be a selection of local federal equilibrium. From Equation (6), as $\beta \rightarrow \infty$, for all n , $x_n^\beta(\Gamma^\beta, \Delta^\beta) - \bar{x}^\beta(\Gamma^\beta, \Delta^\beta) \rightarrow 0$. Because a majority of states must be constrained at any local federal equilibrium, $\Delta^\beta \rightarrow 0$. From Proposition 2, $\theta_\mu \in [\Gamma^\beta - \Delta^\beta, \Gamma^\beta + \Delta^\beta]$, so $\bar{x}^\beta(\Gamma^\beta, \Delta^\beta) \rightarrow \theta_\mu$. From Equation (6), as $\beta \rightarrow \infty$, $x_n^\beta(\Gamma^\beta, \Delta^\beta) = \bar{x}^\beta(\Gamma^\beta, \Delta^\beta) + \frac{\theta_n - \bar{x}^\beta(\Gamma^\beta, \Delta^\beta)}{\beta} + o(\frac{1}{\beta})$. I substitute the previous equality in Equation (1) to obtain

$$\lim_{\beta \rightarrow \infty} V_n^\beta(\Gamma^\beta, \Delta^\beta) = -(\theta_n - \theta_\mu)^2. \tag{32}$$

Suppose to fix ideas that $\theta_\mu < \bar{\theta}$. Equations (31) and (32) imply that the voters of all states $n \leq \mu$ prefer $x(\Gamma^\beta, \Delta^\beta)$ to x^{dec} as $\beta \rightarrow \infty$, which proves part (a). The summation of Equations (31) and (32) over n yields $\lim_{\beta \rightarrow \infty} \sum_n V_n^{\text{dec}, \beta} > \lim_{\beta \rightarrow \infty} \sum_n V_n^\beta(\Gamma^\beta, \Delta^\beta)$, which proves part (b). ■

Proof of Proposition 10. *Step 1: a federal equilibrium exists.* Let $\Delta^e > |\theta_N - \theta_1|$ so that only rightist or only leftist states can be constrained. Given Δ^e , Proposition 2 implies that the Condorcet winner on the Γ dimension is $\Gamma^e = \theta_\mu \pm \Delta^e$. Suppose $\Gamma^e = \theta_\mu + \Delta^e$ to fix ideas. Because $\Gamma^e > \theta_N$, for all $\Delta \geq 0$, rightist states are not constrained and the state equilibrium $x(\Gamma^e, \Delta)$ is equivalent to $x(\Gamma^e + (\Delta^e - \Delta), \Delta^e)$. By construction of Γ^e , it is not majority preferred to $x(\Gamma^e, \Delta^e)$.

Step 2: any federal equilibrium (Γ^e, Δ^e) is majority preferred to decentralization, and $\Delta^e > 0$. Observe that the majority of states always prefer to constrain at least the most extreme states than to constrain none, so $x(\Gamma^e, \Delta^e) \neq x^{\text{dec}}$. For Δ sufficiently large, $x(\Gamma^e, \Delta)$ is equivalent to x^{dec} . By construction, it is not majority preferred to $x(\Gamma^e, \Delta^e)$. Suppose $\Delta^e = 0$, then $\Gamma^e = \theta_\mu$, but then one can easily see from Equation (1) that (Γ^e, ε) is majority preferred to $(\Gamma^e, 0)$ for some $\varepsilon > 0$.

Step 3: proof of part (a). Immediate from Proposition 7.

Step 4: proof of part (b). With the notations of Lemma 3, for almost all Δ ,

$$\begin{aligned} \frac{\partial V_m(\Gamma, \Delta)}{\partial \Delta} &= \frac{\partial [V_m^{\text{dec}}((\max(\underline{t}, \min(\bar{t}, \theta_n)))_n)]}{\partial \Delta} \\ &= \sum_{n > N-r(L, R)} \frac{\partial [V_m^{\text{dec}}(t)]}{\partial t_n} \frac{\partial \bar{t}}{\partial \Delta} + \sum_{n \leq l(L, R)} \frac{\partial [V_m^{\text{dec}}(t)]}{\partial t_n} \frac{\partial \underline{t}}{\partial \Delta}, \\ &= \frac{2\beta}{N(1+\beta)} \left[r(L, R)(\theta_m - R) \frac{\partial \bar{t}}{\partial \Delta} + l(L, R)(\theta_m - L) \frac{\partial \underline{t}}{\partial \Delta} \right]. \end{aligned} \tag{33}$$

From Equation (6), $0 \leq \frac{\partial \bar{x}(L, R)}{\partial R} \leq 1$ so from Equation (9), for almost all Δ , $\frac{\partial \bar{t}}{\partial \Delta} > 0$ and $\frac{\partial \underline{t}}{\partial \Delta} < 0$. Therefore, the term in bracket in the right-hand side of Equation (33) is always negative when $\theta_m \in [\Gamma - \Delta, \Gamma + \Delta]$, which shows that $\frac{\partial V_m(\Gamma, \Delta)}{\partial \Delta}$ is negative for all states m such that $\theta_m \in [\Gamma - \Delta, \Gamma + \Delta]$. Therefore, in the generic case in which all state types are distinct, as $\beta \rightarrow 0$, in any federal equilibrium, at most a bare majority of states are unconstrained, which is clearly socially worse than decentralization.

Step 5: proof of part (c). Suppose that at the federal equilibrium, only rightist states are constrained. Then from Proposition 2, the right bound $\Gamma + \Delta$ is equal to θ_μ , and the left bound is inconsequential. As $\beta \rightarrow \infty$, one can easily see from Proposition 1 that all state policies converge to θ_μ . By symmetry, the same is true if only leftist states are constrained. If both rightist and leftist states are constrained, this means that $\Delta^e \rightarrow 0$ as $\beta \rightarrow \infty$. Therefore, from Proposition 2, $\Gamma^e \rightarrow \theta_\mu$, and $x(\Gamma^e, \Delta^e) \rightarrow (\theta_\mu, \dots, \theta_\mu)$. Finally, the same argument as in the proof of Proposition 9 shows that the federal equilibrium is socially worse than decentralization. ■

Proof of Proposition 11. Suppose for concreteness that citizens vote first on L and then on R . From Lemma 2, the second stage equilibria are the most preferred R of the median state, which exists by continuity of $x(L, R)$.

At the first stage, for any $L \geq \theta_\mu$, one can see from Equation (1) that the most preferred L of the median state is $L = R$. Hence, from Lemma 2, the median states and all leftist states prefer $L = \theta_\mu$ to any $L > \theta_\mu$ at the first stage.

Suppose now that $L < \theta_\mu$. I first show that in this case, the most preferred R of the median state at the second stage are such that $L < R \leq \theta_\mu$. For all L, R such that $L < \theta_\mu \leq R$, the median state is unconstrained at the state equilibrium and from Lemma 3, $V_\mu(L, R) = V_\mu^{\text{dec}}((\max(\underline{t}, \min(\bar{t}, \theta_n)))_n)$ so for almost all R ,

$$\begin{aligned} \frac{\partial V_\mu(L, R)}{\partial R} &= \sum_{n > N-r(L, R)} \frac{\partial [V_\mu^{\text{dec}}(t)]}{\partial t_n} \frac{\partial \bar{t}}{\partial R} + \sum_{n \leq l(L, R)} \frac{\partial [V_\mu^{\text{dec}}(t)]}{\partial t_n} \frac{\partial \underline{t}}{\partial R} \\ &= \frac{2\beta}{N(1+\beta)} \left[r(L, R)(\theta_\mu - R) \frac{\partial \bar{t}}{\partial R} + l(L, R)(\theta_\mu - L) \frac{\partial \underline{t}}{\partial R} \right]. \end{aligned} \quad (34)$$

From Equation (6), $0 \leq \frac{\partial \bar{x}(L, R)}{\partial R} \leq 1$ so from Equation (9), for almost all R , $\frac{\partial \bar{t}}{\partial R} > 0$ and $\frac{\partial \underline{t}}{\partial R} < 0$. Therefore, if $L < \theta_\mu$, for almost all $R \geq \theta_\mu$, $\partial V_\mu / \partial R \leq 0$. Hence, given $L < \theta_\mu$, the most preferred R of the median state is such that $R \leq \theta_\mu$. Clearly, the median state prefers $L = R = \theta_\mu$ to $L \leq R \leq \theta_\mu$, and from Lemma 2, so do all rightist states.

In the simultaneous case, from Lemma 3, the median voters are pivotal on L and R so for any $(L, R) \neq (\theta_\mu, \theta_\mu)$, a change of either L or R can make them better off. Proposition 2 in Loeper (2011) implies that the resulting uniform policy is socially worse than decentralization. ■

Proof of Proposition 12. Let (i_1^e, i_2^e) be a federal equilibrium of (I, f) , and let $(L, R) \in F$. Because f is onto, there exists $(i_1, i_2) \in I$ such that $f(i_1, i_2) = (L, R)$. If i_2 is a subgame equilibrium following i_1 , then because (i_1^e, i_2^e) is a federal equilibrium, $f(i_1^e, i_2^e)$ is majority preferred to $f(i_1, i_2)$, so $f(i_1, i_2)$ does not cover $f(i_1^e, i_2^e)$. Suppose now that i_2 is not a subgame equilibrium following i_1 . Let i_2' be a subgame equilibrium following i_1 . Then $f(i_1, i_2')$ is majority preferred to $f(i_1, i_2)$, and because (i_1^e, i_2^e) is a federal equilibrium, $f(i_1^e, i_2^e)$ is majority preferred to $f(i_1, i_2')$. Therefore, the implication in Definition 4 does not hold, and $f(i_1, i_2)$ does not cover $f(i_1^e, i_2^e)$. ■

Proof of Proposition 13. For the sake of brevity, I use the symbol \succeq for the majority preference relation throughout this proof. I first deal with case (i). Fix β and let $(\theta^k)_{k \in \mathbb{N}}$ be such that $\frac{|\theta_\mu^k - \theta_1^k|}{|\theta_\mu^k - \theta_N^k|} \rightarrow \infty$. W.l.o.g., I renormalize the policy and the type space so that θ_μ^k and θ_1^k are constant and $|\theta_\mu^k - \theta_N^k| \rightarrow 0$.

Step 1: if $(L^k, R^k)_{k \in \mathbb{N}}$ and $(L'^k, R'^k)_{k \in \mathbb{N}}$ are such that $L^k \rightarrow \theta_\mu$ and for all $k \in \mathbb{N}$, $(L'^k, R'^k) \succeq^{\theta^k} (L^k, R^k)$, then $L'^k \rightarrow \theta_\mu$. Because $L^k \rightarrow \theta_\mu$ and $|\theta_\mu^k - \theta_N^k| \rightarrow 0$, $x^{\theta^k}(L^k, R^k) \rightarrow (\theta_\mu, \dots, \theta_\mu)$ so for all $n \geq \mu$, $V_n^{\theta^k}(L^k, R^k) \rightarrow 0$. Because $(L'^k, R'^k) \succeq^{\theta^k} (L^k, R^k)$, for some $p \geq \mu$, $V_p^{\theta^k}(L'^k, R'^k) \geq V_p^{\theta^k}(L^k, R^k)$ so necessarily, $V_p^{\theta^k}(L'^k, R'^k) \rightarrow 0$. Because θ_1^k is bounded away from θ_μ^k , this implies that $L'^k \rightarrow \theta_\mu$.

Step 2: if $(L'^k, R'^k)_{k \in \mathbb{N}}$ and $(L''^k, R''^k)_{k \in \mathbb{N}}$ are such that for all $k \in \mathbb{N}$, $(L''^k, R''^k) \succeq^{\theta^k} (L'^k, R'^k)$ and $(L'^k, R'^k) \succeq^{\theta^k} (\theta_\mu, \theta_\mu)$, then $L''^k \rightarrow \theta_\mu$. From Step 1, $L'^k \rightarrow \theta_\mu$. I apply Step 1 again to (L''^k, R''^k) and (L'^k, R'^k) to get $L''^k \rightarrow \theta_\mu$.

Step 3: if $(L''^k, R''^k)_{k \in \mathbb{N}}$ is such that for all $k \in \mathbb{N}$, (L''^k, R''^k) is uncovered for the preference profile θ^k , then for all $n \geq \mu$, $x^{\theta^k}(L''^k, R''^k) \rightarrow (\theta_\mu, \dots, \theta_\mu)$ and $V_n^{\theta^k}(L''^k, R''^k) \rightarrow 0$. By assumption, (L''^k, R''^k) is not covered by $(L, R) = (\theta_\mu, \theta_\mu)$. This means that either $(L''^k, R''^k) \succeq^{\theta^k} (\theta_\mu, \theta_\mu)$, or $(L''^k, R''^k) \succeq^{\theta^k} (L'^k, R'^k)$ and $(L'^k, R'^k) \succeq^{\theta^k} (\theta_\mu, \theta_\mu)$ for some $(L'^k, R'^k) \in F$. In the first case, Step 1 implies that $L''^k \rightarrow \theta_\mu$. In the second case, Step 2 also implies that $L''^k \rightarrow \theta_\mu$, which proves Step 3.

To conclude the proof of case (i), suppose that for all $k \in \mathbb{N}$, there exists (L''^k, R''^k) which is uncovered for the preference profile θ^k and such that $x^{\theta^k, \text{dec}} \succeq x^{\theta^k}(L''^k, R''^k)$. From Equation (27), for all $n \geq \mu$, $V_n^{\theta^k, \text{dec}} \leq -\frac{\beta}{N} \left(\frac{1}{1+\beta} (\theta_n^k - \theta_1^k) \right)^2$, which is bounded away from 0.

This contradicts Step 3. Therefore, there exists K such that for all $k \geq K$, for the preference profile θ^k , the voters of all states $n \geq \mu$ prefer any uncovered (L, R) to decentralization. Step 3 implies that $x^{\theta^k}(L^k, R^k) \rightarrow (\theta_\mu, \dots, \theta_\mu)$, so the summation of $V_n^{\theta^k}(L^k, R^k)$ over n yields $W^{\theta^k}(L^k, R^k) \leq -N\text{var}(\theta^k) + o(1)$. Together with Equation (27), this shows that $W^{\theta^k, \text{dec}} \geq W^{\theta^k}(L^k, R^k)$ for k sufficiently large.

I now turn to case (ii). Fix θ and consider a sequence $\beta^k \rightarrow \infty$.

Step 0: if $x^{\beta^k}(L^k, R^k)$ converges to some $x \in \mathbb{R}^N$, then $x = (l, \dots, l)$ for some $l \in \mathbb{R}$, and for all $n \in N$, $V_n^{\beta^k}(L^k, R^k) \rightarrow -(l - \theta_n)^2$. Equation (6) implies that

$$x_n^{\beta^k}(L^k, R^k) = \bar{x}^{\beta^k}(L^k, R^k) + \frac{\theta_n - \bar{x}^{\beta^k}(L^k, R^k)}{\beta^k} + o\left(\frac{1}{\beta^k}\right). \quad (35)$$

If $x^{\beta^k}(L^k, R^k) \rightarrow x$, then $\bar{x}^{\beta^k}(L^k, R^k) \rightarrow \bar{x}$, so Equation (35) implies $x^{\beta^k}(L^k, R^k) \rightarrow (\bar{x}, \dots, \bar{x})$. The substitution of Equation (35) in Equation (1) yields $V_n^{\beta^k}(L^k, R^k) \rightarrow -(\bar{x} - \theta_n)^2$.

Step 1: if $(L^k, R^k)_{k \in \mathbb{N}}$ and $(L'^k, R'^k)_{k \in \mathbb{N}}$ are such that $x(L^k, R^k) \rightarrow (\theta_\mu, \dots, \theta_\mu)$ and for all $k \in \mathbb{N}$, $(L'^k, R'^k) \succeq^{\beta^k} (L^k, R^k)$, then $x(L'^k, R'^k) \rightarrow (\theta_\mu, \dots, \theta_\mu)$. Let x be a limit of a subsequence of $x^{\beta^k}(L'^k, R'^k)$. From Step 0, $x = (l, \dots, l)$ for some $l \in \mathbb{R}$, and in the limit, voters' preferences over (L'^k, R'^k) and (L^k, R^k) coincide with their preferences over (l, \dots, l) and $(\theta_\mu, \dots, \theta_\mu)$ when $l \neq \theta_\mu$. The median voter theorem and the hypothesis $(L'^k, R'^k) \succeq^{\beta^k} (L^k, R^k)$ imply then that $l = \theta_\mu$. From the Bolzano Weierstrass theorem, $x^{\beta^k}(L'^k, R'^k)$ must converge to $(\theta_\mu, \dots, \theta_\mu)$.

Step 2: if $(L'^k, R'^k)_{k \in \mathbb{N}}$ and $(L''^k, R''^k)_{k \in \mathbb{N}}$ are such that for all $k \in \mathbb{N}$, $(L''^k, R''^k) \succeq^{\beta^k} (L'^k, R'^k)$ and $(L'^k, R'^k) \succeq^{\beta^k} (\theta_\mu, \theta_\mu)$, then $x(L''^k, R''^k) \rightarrow (\theta_\mu, \dots, \theta_\mu)$. From Step 1, $(L'^k, R'^k) \rightarrow (\theta_\mu, \theta_\mu)$. I apply Step 1 again to (L''^k, R''^k) and (L'^k, R'^k) to get $x(L''^k, R''^k) \rightarrow (\theta_\mu, \dots, \theta_\mu)$.

Step 3: if $(L''^k, R''^k)_{k \in \mathbb{N}}$ is such that for all $k \in \mathbb{N}$, (L''^k, R''^k) is uncovered for β^k , then $x(L''^k, R''^k) \rightarrow (\theta_\mu, \dots, \theta_\mu)$ and for all $n \in N$, $V_n^{\beta^k}(L''^k, R''^k) \rightarrow -(\theta_\mu - \theta_n)^2$. The proof of $x(L''^k, R''^k) \rightarrow (\theta_\mu, \dots, \theta_\mu)$ uses Step 1 and Step 2 exactly as in the proof of case (i). Step 0 implies then that $V_n^{\beta^k}(L''^k, R''^k) \rightarrow -(\theta_\mu - \theta_n)^2$.

To conclude the proof of case (ii), observe that Step 3 is the analog to Equation (32) in the proof of Proposition 9, so the same argument applies. The identity of the supporters of the federal intervention depend on whether $\bar{\theta} \leq \bar{\theta}$ but not on $(L''^k, R''^k)_{k \in \mathbb{N}}$. ■

Proof of Proposition 14. A chain C in F_G is a subset of F_G such that majority preferences are transitive on C . A chain is said to be externally stable if for all $(L, R) \notin F_G$, there exists $(L', R') \in F_G$ that is strictly majority preferred to (L, R) . By definition (Banks 1985), the Banks Set is the set of majority preferred elements of all externally stable chains.

Consider the following subsets of F_G :

$$\begin{aligned} C_L &= \{(L, \theta_\mu): L \in G \text{ and } L \leq \theta_\mu\}, \\ C_R &= \{(\theta_\mu, R): R \in G \text{ and } R \geq \theta_\mu\}. \end{aligned}$$

For any two elements (L, R) and (L', R') of $C_L \cup C_R$, either $L \leq L'$ and $R \leq R'$ or the reverse inequalities hold. Therefore, Lemma 2 implies that on $C_L \cup C_R$, majority preferences coincide with the preferences of the voters of the median state, which are transitive. So $C_L \cup C_R$ is a chain and its majority preferred element is the most preferred alternative of the median voters, which is (θ_μ, θ_μ) .

To show that $C_L \cup C_R$ is externally stable, let $(L, R) \in F_G \setminus (C_L \cup C_R)$. If (θ_μ, θ_μ) is strictly majority preferred to (L, R) , then I am done. Suppose instead that (L, R) is majority preferred to (θ_μ, θ_μ) . Necessarily, $L \leq \theta_\mu \leq R$. To complete the proof, it suffices to show that (θ_μ, R) or (L, θ_μ) is strictly majority preferred to (L, R) . To fix ideas, suppose that $\bar{x}(L, R) \geq \theta_\mu$ (and thus that $\theta_\mu < R$), the proof in the case $\bar{x}(L, R) \leq \theta_\mu$ is symmetric. Because $\theta_N > \theta_\mu$, Proposition 1 implies that $x_N(L, R) > \theta_\mu$, so $r(L, \theta_\mu) > 1$. As shown in the proof of Proposition 11, (L, θ_μ) is preferred to (L, R) by the voters of state μ , and this preference is strict because $r(L, \theta_\mu) > 1$.⁶ From Lemma 2, this implies that (L, θ_μ) is also strictly preferred to (L, R) by the voters of all states $n < \mu$. ■

⁶ To see why the preference must be strict, observe that for all R' close enough to θ_μ , $l(\theta_\mu, R') > 1$. Using the notations in the proof of Proposition 11, for almost all R' , $\frac{\partial \bar{r}}{\partial R}(L, R') > 0$ and $\frac{\partial t}{\partial R}(L, R') < 0$, and since $L < \theta_\mu < R'$, Equation (34) implies that $\frac{\partial V_\mu}{\partial R}(L, R') < 0$.