# "Durable Policy, Political Accountability, and Active Waste"

# Supporting Information

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This document contains the proofs of all Lemmas and Corollaries in the above-mentioned paper.

**Lemma 1** *For risk neutral or risk averse politicians, Pareto efficiency requires that all investment be made in period 1 and at a single poliy; i.e.,* $q_1(p^c) = B$ *for some* $p^c \in \mathbb{R}$*. When politicians are sufficiently risk-seeking investment at two policies is possible if and only if both policies are implemented in equilibrium.*

**Proof of Lemma 1:**

First we show that it is not Pareto efficient for investments to be made at policies that are not implemented. Consider a policy $\tilde{p}$ that is never implemented, and suppose that $q_t^{\tilde{p}} > 0$. The payoff in period $t$ to player $j$ is $-g(p_t - j) + q_t^{p_t}$ where $\tilde{p} \neq p_t$. If the investment at $\tilde{p}$ is instead made at $p_t$, then the payoff to player $j$ would be $-g(p_t - j) + q_t^{p_t} + q_t^{\tilde{p}}$ which is strictly greater than before.

Second we show that when preferences are weakly risk averse, it is never Pareto efficient for investments to be divided among several policies. Consider a history where $\{p_1, \ldots, p_T\}$ is the sequence of implemented policies, and suppose that at least two of these are distinct; i.e. $\exists s, t$ such that $p_s \neq p_t$. Furthermore, suppose that accumulated investment is positive at (at least) two distinct policies; i.e. $\exists s, t$ such that $q_T^{p_s} > 0, q_T^{p_t} > 0$. Consider the payoffs to player $j$ from only periods $s$ and $t$:

$$-g(p_s - j) - g(p_t - j) + q_s^{p_s} + q_t^{p_t}$$

Without loss of generality assume that $s < t$. We show that shifting investments to the mid-point policy of $\bar{p} = \frac{1}{2}(p_s + p_t)$ for period $s$, and implementing $\bar{p}$ in periods $s$ and $t$ is a Pareto improvement. The payoff from periods $s$ and $t$ becomes:

$$-g(\bar{p} - j) + q_s^{p_s} + q_t^{p_t}$$

The payoff to accumulated investment is unchanged, but the ideological payoff is higher as $-g(p_s - j) - g(p_t - j) \leq -g(\bar{p} - j)$ since $g(\cdot)$ is weakly risk averse. Therefore, the resulting shift generates a new history which is a Pareto improvement.

This process of shifting investments and policies in a Pareto improving way can be continued iteratively. Consider the new history, where the implemented policy in periods $s$ and $t$ is $\bar{p}$, consider any policy $p_r$ that is distinct from $\bar{p}$ that remains in the history. Again, we propose shifting investments at $p_r$ and $\bar{p}$ to the mid-point policy $\bar{p}' = \frac{1}{2}(p_r + \bar{p})$ at time period $r$ or $s$ (whichever is earliest). As before, the resulting history investment payoff to all players is (weakly) greater, and so is the ideological payoff, due to risk averse preferences. This Pareto improving process can be iterated until only a single policy is implemented, and all investments are made at the implemented policy in the first period.

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**Corollary 1** *For $g(r-l) < B$, the level of waste in the equilibrium is strictly decreasing in polarization, $|r-l|$, whether polarization is caused by an increase in $r$ or a decrease in $l$.*

**Proof of Corollary 1:** This follows directly from Proposition 1. Since only policy $l$ is implemented in equilibrium, any investment at policy $r$ can be considered waste. In this case waste is equal to $q_1(r)$.

Since $g(\cdot)$ is increasing (in absolute value), $g(r-l)$ is increasing in polarization. When $g(r-l) \leq B$, waste is $\frac{B}{2} - \frac{g(r-l)}{2}$, which is decreasing in $g(r-l)$, and thus in $|r-l|$ by the properties of $g(\cdot)$.

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**Corollary 2** *The median voter's equilibrium per-period utility is:*

$$u_M = \begin{cases} -g(l) + \frac{B}{2} + \frac{g(r-l)}{2} & \text{for} \quad g(r-l) \leq B \\ -g(l) + B & \qquad\quad g(r-l) \geq B. \end{cases}$$

**Proof of Corollary 2:** Follows directly from Proposition 1.

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**Corollary 3** *In equilibrium in the Fixed Policies Regime:*
*(i) $g(r-l) \geq B$ implies $\frac{du_M}{d|l|} < 0$ and $\frac{du_M}{dr} = 0$.*

*(ii)* $g(r-l) \leq B$ *implies* $\frac{du_M}{dr} = \frac{g'(r-l)}{2} > 0$, *and*

$$\frac{du_M}{d|l|} = -g'(-l) + \frac{g'(r-l)}{2} \begin{cases} < 0 & & \textit{risk seeking} \\ < 0 & \textit{when} & \textit{risk neutral} \\ \textit{indeterminate} & & \textit{risk averse} \end{cases}.$$

*(iii) Fixing* $r = -l$, $g(r-l) \leq B$ *implies*

$$\frac{du_M}{d|l|} = -g'(-l) + g'(2l) \begin{cases} < 0 & & \textit{risk seeking} \\ = 0 & \textit{when} & \textit{risk neutral} \\ > 0 & & \textit{risk averse} \end{cases}.$$

**Proof of Corollary 3:**

Case (i): For $g(r-l) \geq B$, $\frac{du_M}{dr} = 0$ is obtained by differentiating $u_M$, which (in that case) is not a function of $r$. Since $g(-x) = g(x)$ for any $x$, and also $l < 0$, then $\frac{du_M}{d|l|} = \frac{du_M}{dl}$ at the point $-l$. But since $l < 0$, then $-l > 0$. So since $g(\cdot)$ is strictly increasing (in absolute value), it must be that $g'(-l) > 0$, which makes $-g'(-l) < 0$.

Case (ii): For $g(r-l) < B$, $\frac{du_M}{dr}$ is obtained by differentiating $u_M$ and since $g(\cdot)$ is strictly increasing (in absolute value), and $r - l > 0$, we have $\frac{g'(r-l)}{2} > 0$. $\frac{du_M}{d|l|}$ is more complicated. The term $-g'(l)$ is obtained as in the previous case. The $\frac{g'(r-l)}{2} > 0$ is found by differentiating, but since the differentiation is with respect to $|l|$, and $l < 0$, the differentiation works as if $l$ were positive, and so the chain-rule does not work in its normal fashion, which is why the term is not multiplied by $-1$.

For $g(r-l) < B$, we have that $\frac{du_M}{d|l|} = -g'(-l) + \frac{g'(r-l)}{2}$. For $g$ strictly convex (risk averse), we know that since $r - l > -l$, then it must be that $g'(r-l) > g'(-l)$. But that doesn't tell us enough to determine the sign of the derivative, since $g'(r-l)$ is multiplied by $\frac{1}{2}$.

For the linear case (risk neutral), for $\frac{du_M}{d|l|} = 0$ we need $2g'(-l) = g'(r-l)$, but we definitely don't have that. By the linearity of $g$, it must be that $g'(-l) = g'(r-l)$, so it cannot be that $2g'(-l) = g'(r-l)$. Given linearity, it must be that in this case the derivative is negative. And that makes intuitive sense, because it in the linear case the voter values ideology and quality equally, and if $l$ moves out further away from the voter, she loses a unit in ideology, but gains only half a unit in quality.

For the concave case (risk seeking), we know that $g'(-l) > g'(r-l) > \frac{g'(r-l)}{2}$. So that implies that $-g'(-l) + \frac{g'(r-l)}{2} < 0$.

Case (iii): Fixing $r = -l$, gives $u_M = -g(l) + \frac{B}{2} + \frac{g(-2l)}{2}$. Differentiating the first term yields $-g'(-l)$ as in case (i). By similar reasoning we also have that $g'(-2l) = g'(-2(-l)) = g'(2l)$. When chain rule is applied it is multiplied by 2, thus giving $\frac{du_M}{d|l|} = -g'(-l) + g'(2l)$.

For $g$ strictly convex (risk averse), we know that $|2l| > |-l|$, so we get $|g'(2l)| > |-g'(-l)|$, which gives $-g'(-l) + g'(2l) > 0$. For $g$ strictly concave (risk seeking), the same reasoning applies but the inequalities are reverse, while for $g$ linear (risk neutral) the inequalities become equalities.

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**Corollary 4** *With quadratic-loss utility, $l^* = r - \sqrt{B}$, and equilibrium is given by three cases:*
*(i) $l \in \left(r - \frac{\sqrt{B}}{2}, 0\right)$. Then $p_1 = 2l - r < l$, with investment levels:*

$$q_1(p_1) = \frac{B}{2} + 2(r-l)^2, \quad q_1(r) = \frac{B}{2} - 2(r-l)^2, \text{ and } q_1(p) = 0 \text{ for all } p \neq p_1, r.$$

*(ii) $l \in \left(l^*, r - \frac{\sqrt{B}}{2}\right)$. Then $p_1 = r - \sqrt{B} < l$, with investment levels $q_1(p_1) = B$, and $q_1(p) = 0$ for all $p \neq p_1$.*
*(iii) $l < l^*$. Then $p_1 = l$ with investment levels $q_1(p_1) = B$, and $q_1(p) = 0$ for all $p \neq p_1$.*

*$L$ wins reelection with certainty, $P(h_1) = 1$. For $t \geq 2$, $p_t = p_1$, with no further investment, and $P(h_t) = 1$.*

**Proof of Corollary 4:** Much of Corollary 4 follows directly from the full statement of Proposition 3. The only element that is not clear is where the boundary $r - \frac{\sqrt{B}}{2}$ comes from in part (ii). This boundary is $\hat{l}$ from the full statement of Proposition 3, and intuitively is the boundary for where investment is wasted at $r$. This boundary occurs when $p_1 = 2l - r = r - \sqrt{B}$. Solving for $l$ from this equation yields $l = r - \frac{\sqrt{B}}{2}$.

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**Corollary 5** *Regardless of risk preferences, the policy implemented in every period satisfies $p_t \leq l$.*

**Proof of Corollary 5:** The intuitive reason why the implemented policy cannot be greater than $l$ is that this makes the policy more appealing to $R$, thus requiring greater wasted investment at $r$ to ensure re-election. The mathematical reason is that the implemented policy must satisfy $-2g'(l-p_t) = -g'(r-p_t)$ in any equilibrium with wasted investment. However, if $p_t > l$, then $-2g'(l-p_t) = 2g'(p_t - l) > 0$ and $-g'(r-p_t) < 0$ (assuming that $p_t < r$). Therefore the equality cannot be satisfied, and a condition of the equilibrium is violated.

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**Corollary 6** *For quadratic-loss utility, the per-period utility for the median voter in equilibrium is:*

$$u_M = \begin{cases} r^2 - 2l^2 + \frac{B}{2} & \text{(i) } l \in \left(r - \frac{\sqrt{B}}{2}, 0\right) \\ -r\left(r - 2\sqrt{B}\right) \quad \text{for} & \text{(ii) } l \in \left(l^*, r - \frac{\sqrt{B}}{2}\right) \\ -l^2 + B & \text{(iii) } l < l^* \end{cases} .$$

**Proof of Corollary 6:** Follows directly from Corollary 4.

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**Corollary 7** *Fixing $r = -l$ for quadratic-loss utility, the cases in Corollary 6 give: (i) $\frac{du_M}{dr} = -2r < 0$, (ii) $\frac{du_M}{dr} = -2r + 2\sqrt{B} > 0$, (iii) $\frac{du_M}{dr} = -2r < 0$.*

**Proof of Corollary 7:**

Case (i): Since $r = -l$, $u_M = -r^2 + \frac{B}{2}$, and differentiating yields $\frac{du_M}{dr} = -2r < 0$

Case (ii): $u_M = -r\left(r - 2\sqrt{B}\right) = -r^2 + 2\sqrt{B}r$, and differentiating yields $\frac{du_M}{dr} = -2r + 2\sqrt{B}$. We also know that, in case(ii), $r < -l^*$, which implies $r < -r + \sqrt{B}$¿ Rearranging we get that $r < \frac{\sqrt{B}}{2}$, which means that $r < \sqrt{B}$. This gives the inequality $\frac{du_M}{dr} = -2r + 2\sqrt{B} > 0$.

Case(iii): Since $r = -l$, $u_M = -r^2 + B$, and differentiating yields $\frac{du_M}{dr} = -2r < 0$

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