Appendix

Lemma 0.1. *In every sequentially rational strategy profile,*

1. *the second-period strategy of officeholder $j \in \{I, C\}$ specifies that j uses skill if $s_j > x_2$ and chooses the default if $x_2 > s_j$;*

2. *the voter's continuation value of electing the challenger is*

$$V(C) = \int_0^1 \left[\int_0^{s_C} s_C dG(x_2) + \int_{s_C}^1 x_2 dG(x_2) \right] dF(s_C), \tag{1}$$

   and $V(C) \in (0, 1)$;

3. *if the incumbent uses skill in the first period, the voter's continuation value of electing the incumbent is*

$$V(I|s_I) = \int_0^{s_I} s_I dG(x_2) + \int_{s_I}^1 x_2 dG(x_2); \tag{2}$$

4. *the voter elects the candidate that provides the greater continuation value.*

Proof. Let σ be a sequentially rational strategy profile.

1. Consider second-period officeholder $j \in \{I, C\}$. Fix $x_2 \in [0, 1]$ and $s_j \in [0, 1]$. Sequential rationality implies j maximizes his second-period payoff. It is immediate that j strictly prefers to use skill if $s_j > x_2$ and strictly prefers the default if $x_2 > s_j$.

2. Part 1 pins down C's strategy in σ outside of the probability zero case $s_C = x_2$. Because R's beliefs about x_2 are represented by G and R's beliefs about s_C are represented by F, Part 1 implies that R's continuation value of electing C under σ is

$$V(C) = \int_0^1 \left[\int_0^{s_C} s_C dG(x_2) + \int_{s_C}^1 x_2 dG(x_2) \right] dF(s_C).$$

Together, the assumptions that g and f are strictly positive over $[0, 1]$ imply $V(C) \in (0, 1)$.

3. Part 1 pins down I's strategy in σ outside of the probability zero case $s_I = x_2$. Given s_I and R's beliefs about x_2, G, Part 1 implies that R's continuation value of electing I under σ is

$$V(I|s_I) = \int_0^{s_I} s_I dG(x_2) + \int_{s_I}^1 x_2 dG(x_2).$$

4. Follows from definition of sequential rationality. $\square$

Lemma 1. *There exists a unique $\bar{s} \in (0,1)$ such that in every sequentially rational strategy profile, $V(I|s_I) > V(C)$ if and only if $s_I > \bar{s}$.*

Proof. Let σ be a sequentially rational strategy profile. By Lemma 0.1, $V(C) \in (0,1)$. Also, both $V(C)$ and $V(I|s_I)$ are constant across sequentially rational strategy profiles. Notice that $V(C)$ is independent of s_I because F and G are both independent of all other features of the game. Also, $V(I|s_I)$ is continuous and strictly increasing in s_I. Furthermore, $V(I|s_I = 0) < V(C) < V(I|s_I = 1)$ because f is strictly positive over $[0,1]$. It follows that there is a unique $\bar{s} \in (0,1)$ such that $V(I|\bar{s}) = V(C)$, $V(I|s_I) < V(C)$ for $s_I < \bar{s}$, and $V(I|s_I) > V(C)$ for $s_I > \bar{s}$. $\square$

Proposition 1. *For all $x_1 \in [0,1]$, every SPE of the complete information model has the following features:*

1. *If $s_I > \bar{s}$ then the voter re-elects the incumbent and if $s_I < \bar{s}$ then the voter elects the challenger.*
2. *Assume $s_I \neq \bar{s}$. If $s_I > x_1$ then the incumbent uses skill, and if $s_I < x_1$ then the incumbent uses the default.*

Proof. Fix $x_1 \in [0,1]$ and let σ be a SPE.
1. Assume $s_I < \bar{s}$. The definition of $\bar{s}$ implies $V(C) > V(I|s_I)$. Sequential rationality of σ requires that R elects C. A symmetric argument implies that R elects I for $s_I > \bar{s}$.
2. Assume $s_I \neq \bar{s}$. First, consider $s_I < x_1$. There are two subcases.

First, assume $s_I < \min\{\bar{s}, x_1\}$. Part 1 implies that R elects C under σ. Sequential rationality requires that I uses skill only if

$$s_I + \beta + \delta V(C) \geq x_1 + \beta + \delta V(C)$$
$$s_I \geq x_1,$$

a contradiction.

Second, assume $s_I \in (\bar{s}, x_1)$. Part 1 implies that R re-elects I under σ. Sequential rationality requires that I uses skill only if

$$s_I + \beta + \delta[V(I|s_I) + \beta] \geq x_1 + \beta + \delta[V(I|s_I) + \beta]$$
$$s_I \geq x_1,$$

a contradiction.

Next, assume $s_I > x_1$. There are two subcases.

First, consider $s_I \in (x_1, \bar{s})$. Part 1 implies that R elects C under σ. Sequential rationality requires that I chooses the default only if

$$\begin{aligned} x_1 + \beta + \delta V(C) &\geq s_I + \beta + \delta V(C) \\ x_1 &\geq s_I, \end{aligned}$$

a contradiction.

Second, consider $s_I > \max\{\bar{s}, x_1\}$. Part 1 implies that R re-elects I under σ. Sequential rationality requires that I chooses the default only if

$$\begin{aligned} x_1 + \beta + \delta[V(I|s_I) + \beta] &\geq s_I + \beta + \delta[V(I|s_I) + \beta] \\ x_1 &\geq s_I, \end{aligned}$$

a contradiction. □

Proposition 2. ***(Low-quality default)*** *If $x_1 \in [0, \bar{s}]$ then there exists an equilibrium that is first-best.*

Proof. There are two cases, $x_1 = 0$ and $x_1 \in (0, \bar{s}]$.

Case 1: $x_1 = 0$

Let $\alpha = (\sigma, \mu)$ be the assessment such that $\mu(s_I; x_1 = 0)$ puts probability one on $s_I = 0$, σ_I^2 and σ_C satisfy Lemma 0.1,

$$\sigma_I^1(s_I; x_1) = \begin{cases} skill & \text{if } s_I > 0 \\ default & \text{if } s_I = 0, \end{cases}$$

and

$$\sigma_R(a_I^1; s_I) = \begin{cases} I & \text{if } s_I > \bar{s} \;\&\; a_I^1 = skill \\ C & \text{else.} \end{cases}$$

The default is observed with probability zero because I uses the default only if $s_I = 0$. Thus, the equilibrium concept places no restrictions on μ if I uses the default in the first period. Because σ_I^2 and σ_C satisfy Lemma 0.1, they satisfy the equilibrium conditions. I now verify that there are no profitable deviations from σ_I^1 and σ_R.

First, consider $s_I > \bar{s}$. Using the default is a profitable deviation for I only if

$$x_1 + \beta + \delta V(C) > s_I + \beta + \delta[V(I|s_I) + \beta] \tag{3}$$

$$V(C) - V(I|s_I) - \beta > s_I - x_1, \tag{4}$$

The definition of $\bar{s}$ implies $V(I|s_I) > V(C)$ for this case. Because $\beta \geq 0$ and $x_1 = 0$, (4) requires $s_I < 0 < \bar{s}$, a contradiction.

Next, consider $s_I \in (0, \bar{s}]$. Using the default is a profitable deviation for I only if

$$x_1 + \beta + \delta V(C) > s_I + \beta + \delta V(C) \tag{5}$$

$$0 > s_I - x_1, \tag{6}$$

a contradiction because $x_1 = 0$ in this case.

Finally, consider $s_I = 0$. Using skill is a profitable deviation for I only if

$$s_I + \beta + \delta V(C) > x_1 + \beta + \delta V(C) \tag{7}$$

$$s_I - x_1 > 0, \tag{8}$$

a contradiction because $x_1 = 0$ in this case.

Together, the three subcases show that I does not have a profitable deviation from σ_I^1.

It follows from Lemma 0.1 that R does not have a profitable deviation from σ_R if I uses skill. If I uses the default, μ places probability one on $s_I = 0 < \bar{s}$. Therefore $V(C) > V(I|\mu)$, and re-electing I is not a profitable deviation. This shows that R does not have a profitable deviation, as desired.

To see that this equilibrium is first-best, notice that I uses skill if $s_I > x_1$ and chooses the default if $s_I < x_1$, and R re-elects I for all $s_I > \bar{s}$ and elects C for all $s_I < \bar{s}$.

Case 2: $x_1 \in (0, \bar{s}]$

Let $\alpha = (\sigma, \mu)$ be the assessment such that σ_I^2 and σ_C satisfy Lemma 0.1,

$$\sigma_I^1(s_I; x_1) = \begin{cases} skill & \text{if } s_I > x_1 \\ default & \text{if } s_I \leq x_1, \end{cases}$$

$$\sigma_R(a_I^1; s_I) = \begin{cases} I & \text{if } s_I > \bar{s} \text{ \& } a_I^1 = \mathit{skill} \\ C & \text{else,} \end{cases}$$

and

$$\mu(s_I; x_1) = \begin{cases} \frac{F(s_I)}{F(x_1)} & \text{for } s_I \in [0, x_1] \\ 1 & \text{for } s_I \in (x_1, 1]. \end{cases}$$

It is straightforward to verify that μ is consistent with σ. Because σ_I^2 and σ_C satisfy Lemma 0.1, they satisfy the equilibrium conditions. I now verify that there are no profitable deviations from σ_I^1 and σ_R.

First, consider $s_I > \bar{s}$. Using the default is a profitable deviation for I only if

$$x_1 + \beta + \delta V(C) > s_I + \beta + \delta[V(I|s_I) + \beta] \tag{9}$$

$$x_1 + V(C) - V(I|s_I) - \beta > s_I, \tag{10}$$

The definition of $\bar{s}$ implies $V(I|s_I) > V(C)$ for this case. Because $\beta \geq 0$, (10) requires $s_I < x_1$, which contradicts $x_1 \leq \bar{s} < s_I$.

Next, consider $s_I \in (x_1, \bar{s}]$. Using the default is a profitable deviation for I only if

$$x_1 + \beta + \delta V(C) > s_I + \beta + \delta V(C) \tag{11}$$

$$x_1 > s_I, \tag{12}$$

a contradiction.

Finally, consider $s_I \leq x_1$. Using skill is a profitable deviation for I only if

$$s_I + \beta + \delta V(C) > x_1 + \beta + \delta V(C) \tag{13}$$

$$s_I > x_1, \tag{14}$$

a contradiction.

Altogether, the three subcases show that I does not have a profitable deviation from σ_I^1.

By Lemma 0.1, R does not have a profitable deviation from σ_R if I uses skill. To see that R does not have a profitable deviation from σ_R if I uses the default, notice that $x_1 \leq \bar{s} < 1$ implies $F(x_1) < 1$ because f is strictly positive over $[0, 1]$. Therefore $\mu(s_I; x_1) = \frac{F(s_I)}{F(x_1)} > F(s_I)$ for all $s_I \in [0, x_1)$. Because $\mu(s_I; x_1) = 1$ for $s_I \geq x_1$ it follows that $\mu(s_I; x_1) \geq F(s_I)$ for $s_I \geq x_1$. Therefore μ is first order stochastically dominated by F. Thus, R strictly prefers to elect C after observing x_1. This shows that R does not have a profitable deviation.

To see that this equilibrium is first-best, notice that I uses skill if $s_I > x_1$ and chooses the default if $s_I < x_1$, and R re-elects I for all $s_I > \bar{s}$ and elects C for all $s_I < \bar{s}$. □

Proposition 3. *There exists $\underline{x}$ such that if $x_1 \in (\underline{x}, \bar{s})$ then every equilibrium is first-best. Additionally, $\underline{x}$ is strictly decreasing in β and there exists $\overline{\beta} > 0$ such that if $\beta > \overline{\beta}$ then every equilibrium is first-best for all $x_1 \in [0, \bar{s}]$.*

Proof. Define $\underline{x} = \delta[V(C) - V(I|s_I = 0) - \beta]$, and consider $x_1 \in (\underline{x}, \bar{s}]$. Clearly, $\underline{x}$ is strictly decreasing in β. Let $\alpha = (\sigma, \mu)$ denote an equilibrium. Because $x_1 \leq \bar{s}$, σ specifies that I uses skill if $s_I > \bar{s}$.

The proof proceeds in two parts. In the first part, I show that if $x_1 \in (\underline{x}, \bar{s}]$ then α specifies that R elects C with probability one if I chooses the default. Using part one, the second part shows that α is first-best.

Part 1: I first show that R must elect C in equilibrium if I chooses the default. To show a contradiction, assume R re-elects I with probability $\eta \in (0, 1]$ if I chooses the default. By Lemma 1, I loses re-election after using skill if $s_I < \bar{s}$. Thus, I strictly prefers to choose the default at $s_I < \bar{s}$ if and only if

$$x_1 + \beta + \delta[(1-\eta)V(C) + \eta(V(I|s_I) + \beta)] > s_I + \beta + \delta V(C). \tag{15}$$

There are two cases: $\underline{x} \geq 0$ and $\underline{x} < 0$.

First, consider $\underline{x} \geq 0$. Notice that

$$x_1 + \beta + \delta[(1-\eta)V(C) + \eta(V(I|s_I = 0) + \beta)] > \underline{x} + \beta + \delta[(1-\eta)V(C) + \eta(V(I|s_I = 0) + \beta)] \tag{16}$$

$$\geq \eta\underline{x} + \beta + \delta[(1-\eta)V(C) + \eta(V(I|s_I = 0) + \beta)], \tag{17}$$

where $x_1 > \underline{x}$ implies (16), and (17) follows from $\eta \in (0, 1]$ for $\underline{x} \geq 0$. Using the definition of $\underline{x}$ and simplifying,

$$\eta\underline{x} + \beta + \delta[(1-\eta)V(C) + \eta(V(I|s_I = 0) + \beta)] = \beta + \delta V(C). \tag{18}$$

Using (18), (17) implies

$$x_1 + \beta + \delta[(1-\eta)V(C) + \eta(V(I|s_I = 0) + \beta)] > s_I + \beta + \delta V(C), \tag{19}$$

for $s_I = 0$. Because both sides of (19) are continuous in s_I, there exists $\underline{s} \in (0, \bar{s})$ such that

$$x_1 + \beta + \delta[(1-\eta)V(C) + \eta(V(I|s_I) + \beta)] > s_I + \beta + \delta V(C) \tag{20}$$

for all $s_I \in [0, \underline{s})$.

Second, consider $\underline{x} < 0$. Notice that $x_1 \geq 0 > \underline{x}$ and $\eta \in (0, 1]$ imply

$$x_1 + \beta + \delta[(1-\eta)V(C) + \eta(V(I|s_I = 0) + \beta)] \geq 0 + \beta + \delta[(1-\eta)V(C) + \eta(V(I|s_I = 0) + \beta)] \tag{21}$$

$$> \eta\underline{x} + \beta + \delta[(1-\eta)V(C) + \eta(V(I|s_I = 0) + \beta)]. \tag{22}$$

An argument analogous to the first case then establishes the existence of $\underline{s} \in (0, \bar{s})$ such that (19) holds for all $s_I \in [0, \underline{s})$.

We have shown α must specify that I chooses the default for all $s_I \in [0, \underline{s})$. Thus, μ is well defined and places positive probability on $[0, \underline{s})$. Because $x_1 < \bar{s}$, we know I uses skill if $s_I > \bar{s}$. Thus, μ places probability zero on s_I such that $V(I|s_I) > V(C)$ and positive probability on s_I such that $V(I|s_I) < V(C)$. It follows that $V(I|\mu) < V(C)$ if I chooses the default under α. Therefore α specifies that I elects C with probability one if I chooses the default, a contradiction.

Part 2: The preceding argument establishes that R elects C with probability one in every equilibrium if $x_1 \in (\underline{x}, \bar{s}]$. I now show that this implies every equilibrium is first-best.

We know σ specifies that I use skill if $s_I > \bar{s}$. Consider $s_I < \bar{s}$. The condition for I to strictly prefer to use skill under α is

$$s_I + \beta + \delta V(C) > x_1 + \beta + \delta V(C) \tag{23}$$

$$s_I > x_1. \tag{24}$$

It follows that I strictly prefers to use skill for all $s_I \in (x_1, \bar{s})$ and strictly prefers to choose the default for all $s_I \in [0, x_1)$.

Finally consider $s_I = \bar{s}$ and again let η be the probability that R re-elects I after observing $s_I = \bar{s}$. The condition for I to strictly prefer to use skill under α is

$$\bar{s} + \beta + \delta[(1-\eta)V(C) + \eta(V(I|s_I = \bar{s}) + \beta)] > x_1 + \beta + \delta V(C), \tag{25}$$

which is equivalent to

$$\begin{aligned} \bar{s} + \delta\eta[V(I|s_I = \bar{s}) - V(C) + \beta] &> \bar{s} + \delta\eta\beta & (26) \\ &\geq x_1, & (27) \end{aligned}$$

where (26) follows from $V(I|s_I = \bar{s}) = V(C)$ and (27) follows from $\delta\eta\beta \geq 0$. Thus, I strictly prefers to use skill in this case if $x_1 < \bar{s}$.

Altogether, we have shown that I uses skill if $s_I > x_1$ and chooses default if $s_I < x_1$ under α. Thus, α is such that R re-elects I for all s_I such that $V(I|s_I) > V(C)$ and elects C for all s_I such that $V(I|s_I) < V(C)$. This establishes that α is first-best, as desired. □

Proposition 4. *If $x_1 \in [0, \bar{s}]$ then ex ante there is zero probability that the voter re-elects the incumbent for choosing the default in equilibrium.*

Proof. Consider $x_1 \in [0, \bar{s}]$. Let $\alpha = (\sigma, \mu)$ denote an equilibrium. Because $x_1 \leq \bar{s}$, α specifies that I use skill if $s_I > \bar{s}$. Therefore I must choose the default with probability zero under α in order for R to re-elect I with positive probability after observing x_1. It follows that ex ante there is zero probability of observing I win re-election after choosing the default under α. □

Proposition 5. *If $x_1 \in (\bar{s}, 1)$ then every equilibrium of the incomplete information model has the following features:*

1. *If the incumbent chooses the default, or if $s_I < \bar{s}$ and the incumbent uses skill, then the voter elects the challenger. If $s_I > \bar{s}$ and the incumbent uses skill, then the voter re-elects the incumbent.*

2. *There exists $s_\beta \in [\bar{s}, x_1)$ such that the incumbent uses skill if $s_I > s_\beta$, and chooses the default if $s_I < s_\beta$.*

Proof. Fix $x_1 \in (\bar{s}, 1)$ and let $\alpha = (\sigma, \mu)$ be an equilibrium.

1. Because σ is sequentially rational, Lemma 0.1 implies that σ_R must specify that R elects I if $s_I > \bar{s}$ and I uses skill and elects C if $s_I < \bar{s}$ and I uses skill.

I now prove that α must specify that R elects C after observing the default. To show a contradiction, assume that α specifies that R elects I after observing the default. Because α is an equilibrium, I's strategy must be sequentially rational. I now show that this implies that I uses skill if $s_I > x_1$ and uses the default if $s_I < x_1$.

First, consider $s_I > x_1$. Sequential rationality requires that I chooses the default only if

$$\begin{aligned} x_1 + \beta + \delta[V(I|s_I) + \beta] &\geq s_I + \beta + \delta[V(I|s_I) + \beta] \\ x_1 &\geq s_I, \end{aligned}$$

a contradiction. Thus, α specifies that I uses skill.

Next, consider $s_I \in (\bar{s}, x_1)$. Sequential rationality requires that I uses skill only if

$$\begin{aligned} s_I + \beta + \delta[V(I|s_I) + \beta] &\geq x_1 + \beta + \delta[V(I|s_I) + \beta] \\ s_I &\geq x_1, \end{aligned}$$

a contradiction. Thus, α specifies that I chooses the default.

Consider $s_I = \bar{s}$. By definition, $V(I|\bar{s}) = V(C)$. Sequential rationality requires that I uses skill only if

$$\bar{s} + \beta + \delta V(C) \geq x_1 + \beta + \delta[V(I|\bar{s}) + \beta]. \tag{28}$$

By $x_1 > \bar{s}$ and $\beta \geq 0$,

$$\begin{aligned} x_1 + \beta + \delta[V(I|\bar{s}) + \beta] &> \bar{s} + \beta + \delta[V(I|\bar{s}) + \beta] \qquad (29) \\ &\geq \bar{s} + \beta + \delta V(C), \qquad (30) \end{aligned}$$

which contradicts (28). Thus, α specifies that I chooses the default.

Finally, consider $s_I \in [0, \bar{s})$. By (29) and (30),

$$\begin{aligned} x_1 + \beta + \delta[V(I|\bar{s}) + \beta] &> \bar{s} + \beta + \delta V(C) \\ x_1 &> \bar{s} + \delta[V(C) - V(I|\bar{s}) - \beta] \qquad (31) \end{aligned}$$

is satisfied for $\bar{s}$. To show that (31) is satisfied for s_I, I prove that the right hand side

(RHS) of (31) is strictly increasing in s_I.

By Lemma 0.1,

$$V(I|s_I) = \int_0^{s_I} s_I dG(x_2) + \int_{s_I}^1 x_2 dG(x_2). \tag{32}$$

Define $\tilde{G}(a) = \int_0^a G(x_2)dx_2$. Applying integration by parts to (32) yields

$$\begin{aligned}\int_0^{s_I} s_I dG(x_2) + \int_{s_I}^1 x_2 dG(x_2) &= s_I G(s_I) - s_I G(s_I) + G(1) - \int_{s_I}^1 G(x_2)dx_2 \\ &= G(1) - \tilde{G}(1) + \tilde{G}(s_I).\end{aligned}$$

Thus, the RHS of (31) is equivalent to

$$s_I + \delta[V(C) - G(1) + \tilde{G}(1) - \tilde{G}(s_I) - \beta]. \tag{33}$$

Applying the Fundamental Theorem of Calculus to $\tilde{G}(s_I)$, the partial derivative of (33) with respect to s_I is $1 - \delta G(s_I)$. Together, $\delta \in (0, 1]$ and g strictly positive over $[0, 1]$ imply $1 - \delta G(s_I) > 0$ for $s_I < 1$, so (33) is strictly increasing in s_I. Because $s_I = \bar{s}$ satisfies (31), all $s_I \in [0, \bar{s})$ satisfy (31). Therefore, I chooses the default if $s_I \in [0, \bar{s})$.

I have shown that if R elects I after observing the default, then α must specify that I uses skill if $s_I \in (x_1, 1]$ and chooses the default if $s_I \in [0, x_1)$. Consistency of α requires that R's beliefs about s_I after observing the default are $\mu(s_I; x_1) = \frac{F(s_I)}{F(x_1)} > F(s_I)$ for all $s_I \in [0, x_1)$, where the inequality follows from $F(x_1) < 1$ because $x_1 < 1$ and f is strictly positive over $[0, 1]$. Because $\mu(s_I; x_1) = 1$ for $s_I \geq x_1$ it follows that $\mu(s_I; x_1) \geq F(s_I)$ for $s_I \geq x_1$. Thus, μ is first order stochastically dominated by F. It follows that R has a profitable deviation to elect C after observing x_1, a contradiction.

2. Define $\hat{s}_\beta$ to be the unique $s \in \mathbb{R}$ that solves

$$s + \delta V(I|s) = x_1 + \delta[V(C) - \beta]. \tag{34}$$

To see that $\hat{s}_\beta$ exists, notice that the left hand side of (34) is continuous and strictly increasing in s and the right hand side is constant in s. Notice that (34) is equivalent to

$$s = x_1 + \delta[V(C) - V(I|s) - \beta]. \tag{35}$$

It follows that

$$s < x_1 + \delta[V(C) - V(I|s) - \beta] \tag{36}$$

if and only if $s < \hat{s}_\beta$ and

$$s > x_1 + \delta[V(C) - V(I|s) - \beta] \tag{37}$$

if and only if $s > \hat{s}_\beta$. Finally, inspection of (35) shows that $\hat{s}_\beta < x_1$ because $x_1 > \bar{s}$, $\delta > 0$, $\beta \geq 0$, and $V(I|s) > V(C)$ for $s > \bar{s}$.

Let $s_\beta = \max\{\bar{s}, \hat{s}_\beta\}$. Clearly, $s_\beta \geq \bar{s}$ by definition. Also, properties of $\hat{s}_\beta$ imply $s_\beta < x_1$. Thus, $s_\beta \in [\bar{s}, x_1)$

Consider $s_I > s_\beta$. Sequential rationality requires that I uses skill at s_I if

$$\begin{aligned} s_I + \beta + \delta[V(I|s_I) + \beta] &> x_1 + \beta + \delta V(C) \\ s_I &> x_1 + \delta[V(C) - V(I|s_I) - \beta], \end{aligned} \tag{38}$$

which is equivalent to $s_I > \hat{s}_\beta$ by (37). Because $s_I > s_\beta \geq \hat{s}_\beta$, (38) is satisfied.

Next, assume $s_I < s_\beta$. There are three subcases.

First, consider $s_I \in (\bar{s}, s_\beta)$. Sequential rationality requires that I uses the default at s_I if

$$s_I + \beta + \delta[V(I|s_I) + \beta] < x_1 + \beta + \delta V(C) \tag{39}$$

$$s_I < x_1 + \delta[V(C) - V(I|s_I) - \beta], \tag{40}$$

which is equivalent to $s_I < \hat{s}_\beta$ by (36). By definition, $s_\beta > \bar{s}$ requires $s_\beta = \hat{s}_\beta$, so (40) is equivalent to $s_I < s_\beta$, which holds.

Next, consider $s_I = \bar{s} < s_\beta$. Notice that $\beta \geq 0$ and the definition of $\bar{s}$ imply

$$s_I + \beta + \delta[V(I|s_I) + \beta] \geq s_I + \beta + \delta V(C). \tag{41}$$

Therefore I weakly prefers to win re-election after using skill if $s_I = \bar{s}$. Thus, sequential rationality requires that I uses the default at s_I if

$$s_I + \beta + \delta[V(I|s_I) + \beta] < x_1 + \beta + \delta V(C). \tag{42}$$

Notice that (42) is equivalent to (39). Because $s_I = \bar{s} < s_\beta$, (36) implies that (42) holds.

Thus, α must specify that I uses the default at s_I.

Finally, consider $s_I < \bar{s}$. Sequential rationality requires that I uses the default at s_I if

$$s_I + \beta + \delta V(C) < x_1 + \beta + \delta V(C)$$
$$s_I < x_1,$$

which holds because $s_I < \bar{s} \leq s_\beta < x_1$.

Altogether, the three cases establish that α must specify that I uses the default if $s_I < s_\beta$, as desired. □

Lemma 2. *If $x_1 \in (\bar{s}, 1)$ then there exists $s_0 \in [s_\beta, x_1)$ such that in every equilibrium efficient showing off occurs if $s_I \in [s_0, x_1)$ and inefficient showing off occurs if $s_I \in (s_\beta, s_0)$.*

Proof. Let $\alpha = (\sigma, \mu)$ be an equilibrium.

As in Proposition 5, define $s_\beta = \max\{\bar{s}, \hat{s}_\beta\}$, where $\hat{s}_\beta$ is the unique $s \in \mathbb{R}$ that solves

$$s + \delta V(I|s) = x_1 + \delta[V(C) - \beta]. \tag{43}$$

The right hand side of (43) is strictly decreasing in β and constant in s, while the left hand side of (43) is strictly increasing in s and constant in β. Therefore, $\hat{s}_\beta$ is strictly decreasing in β.

Because $x_1 > \bar{s}$ and $\delta > 0$, (43) implies $\hat{s}_0 > \bar{s}$, where $\hat{s}_0$ is $\hat{s}_{\beta=0}$. It follows that $s_0 = \hat{s}_0$. By Proposition 5, $s_0 < x_1$ and I shows off at $s_I \in (s_\beta, x_1)$ under α. If $s_I \in [s_0, x_1)$ then $s_I + \delta V(I|s_I) \geq x_1 + \delta V(C)$, so I using skill and winning re-election is second-best. Thus, showing off is efficient in this case. On the other hand, if $s_I \in (s_\beta, s_0)$ then $s_I + \delta V(I|s_I) < x_1 + \delta V(C)$, so I using skill and winning re-election is not second best, and showing off is inefficient. □

Proposition 6. *Assume $x_1 \in (\bar{s}, 1)$. If $\beta \in [0, \frac{x_1 - \bar{s}}{\delta})$ then in every equilibrium the occurrence of inefficient showing off is strictly increasing in β. If $\beta \geq \frac{x_1 - \bar{s}}{\delta}$ then in every equilibrium the incumbent uses skill in the first period for all $s_I \in (\bar{s}, 1]$.*

Proof. Fix $x_1 \in (\bar{s}, 1)$. Let $\alpha = (\sigma, \mu)$ be an equilibrium. Define $\hat{s}_\beta$ as in Proposition 5 and let $s_\beta = \max\{\bar{s}, \hat{s}_\beta\}$.

I first show that s_β is strictly decreasing in β for $\beta \in [0, \frac{x_1 - \bar{s}}{\delta})$. Recall that $\hat{s}_0 > \bar{s}$, $\hat{s}_\beta$ is continuous and strictly decreasing in β, and $\bar{s}$ is constant in β. Thus, there exists

$\bar{\beta} > 0$ such that $s_\beta = \hat{s}_\beta > \bar{s}$ if $\beta \in [0, \bar{\beta})$ and $s_\beta = \bar{s}$ if $\beta \geq \bar{\beta}$. In particular,

$$\bar{s} + \bar{\beta} + \delta[V(I|\bar{s}) + \bar{\beta}] = x_1 + \beta + \delta V(C) \tag{44}$$

$$\bar{\beta} = \frac{x_1 - \bar{s}}{\delta} + V(C) - V(I|\bar{s}) \tag{45}$$

$$\bar{\beta} = \frac{x_1 - \bar{s}}{\delta}, \tag{46}$$

where (46) follows from (45) because $V(C) = V(I|\bar{s})$ by definition of $\bar{s}$.

Assume $\beta \in [0, \bar{\beta})$. Then $s_\beta = \hat{s}_\beta$ and it follows that s_β is strictly decreasing in β. By Lemma 2, inefficient showing off occurs at $s_I \in (s_\beta, s_0)$. Because $s_\beta = \hat{s}_\beta > \bar{s}$, s_β is strictly decreasing in β. Recall that s_0 is constant in β. Therefore the occurrence of inefficient showing off is strictly increasing in β.

To see that I uses skill at all $s_I \in (\bar{s}, 1]$ if $\beta \geq \bar{\beta}$, notice that (46) implies $s_\beta = \bar{s}$ for such β. By Proposition 5, I uses skill if $s_I > s_\beta = \bar{s}$, as desired. □

Proposition 7. *There exists $s^* \in [\bar{s}, 1)$ such that if $x_1 \in (s^*, 1)$ then showing off occurs in every equilibrium.*

Proof. Let $\alpha = (\sigma, \mu)$ denote an equilibrium. Define $s^* = \max\{\bar{s}, \frac{1+\delta V_I(C|H)}{1+\delta}\}$. Notice that $V(C|H) < 1$ because f is strictly positive over $s_C \in [0, 1]$ and H is not degenerate on $x_2 = 1$. Thus, $\frac{1+\delta V_I(C|H)}{1+\delta} < 1$ and $\bar{s} < 1$, so $s^* < 1$. Consider $x_1 \in (s^*, 1)$. Because $s^* \geq \bar{s}$, we know that I uses skill if $s_I > x_1$.

Assume that showing off does not occur under α. Consistency of μ requires that R's beliefs about s_I after observing the default under α are $\mu(s_I; x_1) = \frac{F(s_I)}{F(x_1)} > F(s_I)$ for all $s_I \in [0, x_1)$, where the inequality follows from $F(x_1) < 1$ because $x_1 < 1$ and f is strictly positive over $[0, 1]$. Because $\mu(s_I; x_1) = 1$ for $s_I \geq x_1$ it follows that $\mu(s_I; x_1) \geq F(s_I)$ for $s_I \geq x_1$. Thus, μ is first order stochastically dominated by F, so α must specify that R elects C if I chooses the default.

Consider $s_I \in (s^*, x_1)$. Because $s_I \geq \bar{s}$, R re-elects I if I uses skill. Notice that

$$s_I + \beta + \delta[V(I|s_I, x_1, a_I^1 = s_I) + \beta] \geq s_I + \beta + \delta(s_I + \beta) \tag{47}$$

$$= (1+\delta)(s_I + \beta) \tag{48}$$

$$> 1 + \delta V(C|H) + \beta \tag{49}$$

$$> x_1 + \beta + \delta V(C|s_I, x_1, a_I^1 = x_1), \tag{50}$$

where (47) follows from $s_I \leq V(I|s_I, x_1, a_I^1 = s_I)$, (49) from $s_I > \frac{1+\delta V(C|H)}{1+\delta}$ and $\beta \geq 0$, and (50) from $V(C|H) > V(C|s_I, x_1, a_I^1)$ for all s_I, x_1, and a_I^1. This establishes that I

has a profitable deviation to use skill at s_I, contradicting the assumption that α is an equilibrium. $\square$

Proposition 8. *If $x_1 \in int(S^W)$ then there does not exist an equilibrium that is first-best.*

Proof. Let $\alpha = (\sigma, \mu)$ be an equilibrium that is first-best. Consider $x_1 \in \text{int}(S^W)$. Define $\hat{a}(s_I, x_1) \in [0,1]^N$ to be the N-dimensional vector such that $\hat{a}_n(s_I, x_1) = \max\{s_I^n, x_1^n\}$ for each $n \in N$.

Because α is first-best, $\sigma_I^1 = \hat{a}(s_I, x_1)$ under α. If I chooses the default on every issue, consistency of μ and independence of F_n across n imply

$$\mu_n(s_I^n; x_1^n) = \begin{cases} \frac{F_n(s_I^n)}{F_n(x_1^n)} & \text{for } s_I^n \in [0, x_1^n) \\ 1 & \text{for } s_I^n \in [x_1^n, 1], \end{cases} \tag{51}$$

for all $n \in N$. It follows that $\mu_n(s_I^n; x_1^n) > F_n(s_I^n)$ for $s_I^n \in [0, x_1^n)$ because $x_1 \in \text{int}(S^W)$ implies $x_1^n < 1$ and f_n is strictly positive over $[0,1]$, so $F_n(x_1^n) < 1$. Additionally, $\mu_n(s_I^n; x_1^n) = 1$ for $s_I^n \geq x_1^n$. Thus, $\mu_n(s_I^n; x_1^n) \geq F_n(s_I^n)$ for $s_I^n \geq x_1^n$. It follows that μ_n is first order stochastically dominated by F_n for all $n \in N$. Therefore, if I chooses the default on every policy issue then R strictly prefers C on every dimension and strictly prefers to elect C. By sequential rationality, α must specify that R elects C if I chooses default policy on every issue, i.e. $\hat{a}(s_I, x_1) = x_1$.

By $x_1 \in \text{int}(S^W)$, there exist $s_I \in S^W$ such that $s_I^n < x_1^n$ for all $n \in N$. Consider such s_I. Because α is first-best, $\hat{a}_n(s_I, x_1) = x_n^1$. By $s_I \in S^W$, I wins re-election by using skill on every dimension. Note that $s_I^n < x_1^n$ for all $n \in N$ implies that s_I is the worst possible policy that guarantees I re-election. Deviating to s_I is profitable for I if and only if

$$\begin{aligned} \omega \cdot s_I + \beta + \delta[V(I|s_I) + \beta] &> \omega \cdot x_1 + \beta + \delta V(C) \\ \omega \cdot (s_I - x_1) &> \delta[V(C) - V(I|s_I) - \beta]. \end{aligned} \tag{52}$$

Because $s_I \in S^W$, $V(I|s_I) > V(C)$, which implies that the right hand side of (52) is strictly negative because $\delta > 0$ and $\beta \geq 0$. Because $x_1 \in \text{int}(S^W)$ there exist s_I for which (52) is satisfied, a contradiction. $\square$