



## KNUT WICKSELL ON THE OPTIMAL ROTATION PROBLEM IN FORESTRY

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### ABSTRACT

*This paper introduces the ideas related to economically optimal forest management, which are contained in a recent discovered and published manuscript written by the Swedish economist Knut Wicksell. The manuscript shows a slightly different approach to capital and investment theory, than the one that has reigned since Irving Fisher's "The Theory of Interest Rates". Moreover, Wicksell solves, given his approach, the optimal rotation problem in forestry in an appropriate manner.*

*Keywords: Optimal rotation, Faustmann solution, endogenous interest rate.*



### INTRODUCTION

This paper introduces the ideas related to economically optimal forest management, which are contained in a recently discovered and published manuscript written by the Swedish economist Knut Wicksell<sup>1</sup>. The manuscript is interesting for at least two reasons. Firstly, it represents a slightly different approach to capital and investment theory, than the one that has reigned since Irving Fisher's "The Theory of Interest Rates" (1930), which contains his famous separation theorem; in a perfect capital market optimal investment and consumption decisions can be made independently of each other. Secondly, unlike Fisher and many other economists, Wicksell solves, given his approach, the optimal rotation problem in the appropriate manner.

The paper is structured as follows: It starts with a brief rhapsodic account of Wicksell's life and his contributions to economic theory. We then turn to a special detail in the Norwegian Nobel Laureate, Ragnar Frisch's, evaluation of the researcher Wicksell. This is followed by the presentation of Wicksell's ideas, and a few concluding comments.

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<sup>1</sup> See Wicksell (1987), and Torun Hedlund-Nyström et al. (1993).

## KNUT WICKSELL AS AN ECONOMIST

Johan Gustaf Knut Wicksell (1851–1926) stands out among the early 20:th century Scandinavian school of economists, which includes such names as Gustav Cassel, Eli F. Heckscher, Erik Lindahl, Gunnar Myrdal, and Bertil Ohlin in Sweden; Ragnar Frisch in Norway; and Frederik Zeuthen in Denmark. His principal contributions to economics are his development of the marginal productivity theory of distribution, integrating it with the theory of capital and interest, and his original ideas on monetary theory. The latter is based on the notion of monetary equilibrium, and on the distinction between the actual interest rate, and the “natural” one, which would equate the amount of loan capital demanded and that of savings supplied.

He started as a mathematician, but his post graduate university studies were not regular — he worked occasionally as a teacher — so that it took him thirteen years to finish his final university examination in mathematics (*Licentiatius philosophiae*) in 1885. He was later to take two degrees in economics; *viva voce* examination in 1893, which added to his *Licentiate* degree in mathematics, and a Ph.D. in economics in 1895. He also had to take an undergraduate degree in law (1899). The latter was necessary to obtain a position in economics, since economics at that time in Sweden was housed within the Faculty of Law. He became Professor of Economics and Public Law at the University of Lund in 1901 after a fierce battle with Gustav Cassel. Cassel, however, felt that political considerations were used to Wicksell’s disadvantage and his sense of fair play induced him to withdraw his application before the final decision.

Wicksell was a neo-malthusian and advocated open access to contraceptives. In his youth he was in contact with radicals such as the writer August Strindberg, and the first Social Democratic Prime Minister, Hjalmar Branting. In November 1908 Wicksell gave a talk in Stockholm on “The throne, the altar, the sword and the purse”, and he obviously expressed himself in a “too open way”, since he was sentenced to two months of imprisonment.

After his retirement in 1916 he and his wife Anna Bugge (they were never formally married) moved back to Stockholm, where he was born. He participated frequently in

the meeting of the Swedish Economic Association, and two years before his death he made a comment on a speech delivered by Harald Westergård on "Economic Barometers". The statement is available in English in Frisch (1952), but I cannot resist to cite a small part from it, since I think the underlying ideas are relevant for today's two competing views on the business cycle; the endogenous business cycle, and the real business cycle.

*"I have many times used the analogy that if one hits a rocking horse with a hammer, the blows may fall quite irregularly and still the movements of the rocking horse be more or less regular because of its own form."*

Wicksell died from pneumonia on the 3rd of May 1926.

Knut Wicksell's principal works are available in English translations: *Interest and Prices* (1936; first published 1898 in German; *Geldzins und Güterpreise*) and *Lectures on Political Economy* (2 vols; 1934, first published 1901–1906 in Swedish; *Lektioner i nationalekonomi*). Two other books are available in German. They are *"Über Wert Kapital und Rente"* (1893) and *"Finanztheoretische Untersuchungen"* (1896). The latter, which is based on Wicksell's Ph. D. thesis in Economics, is the fundament on which Erik Lindahl builds his path breaking Ph. D. thesis *"Die Gerechtigkeit der Besteuerung"* (1919). Today you cannot open a good textbook in Public Economics without coming across terms such as Lindahl equilibria and Lindahl prices. These concepts have today natural applications in resource and environmental economics, since many environmental goods are public goods.

#### FRISCH ON WICKSELL

Ragnar Frisch, in a review of Knut Wicksell's contributions to economic theory, once wrote:

*"Sometimes it happened that I thought I had finally caught him in an inconsistency or in unclear thinking. Every time this happened, it turned out, however, that the error was mine. After a number of such experiences, I reached the conclusion that whenever a person thinks that he has found an inconsistency or a*

*piece of unclear thinking in Wicksell's works, and wants to "correct" it, that is only one sure criterion that the person in question has not yet penetrated to the bottom of Wicksell's ideas".<sup>2</sup>*

I have not been a frequent reader of Wicksell, but I have certainly made the same observation as Frisch. The latest time this happened was during a sabbatical in Berkeley 1987, when I was reviewing candidates for a chair in macroeconomics at the Stockholm School of Economics. One of the applicants (Lars Jonung) had submitted a comment on a previously unpublished paper by Knut Wicksell<sup>3</sup>. He and his coauthors were mainly interested in dating the manuscript, but what caught my eyes was that Wicksell started his analysis of a flow input-point output problem by explicitly treating the optimal rotation problem in forestry. (In his Lectures<sup>4</sup> he only treats wine aging with a reference to Jevons.) To my "surprise" he already in the second equation came up with, as I thought, an incorrect formula for the determination of the optimal cutting age of a forest stand. I was at the time teaching a course in resource economics, and I told my students, "en passant", that even Wicksell went wrong dealing with the optimal rotation problem<sup>5</sup>.

After a while, in the back of my head, I remembered Frisch's words, which had been brought to my attention by Carl G. Uhr, when he taught a graduate course on Wicksell in the mid-sixties in Umeå! I rethought the analysis, and, of course, it turned out that I had "not penetrated to the bottom of Wicksell's ideas". Wicksell was right; at least as right as he is about the wine aging problem in his Lectures. To understand the issues involved we have to turn to a more formal analysis.

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<sup>2</sup> See Frisch (1952).

<sup>3</sup> Later published in Ekonomiska Samfundets Tidskrift, No. 3, 1987, as "Ett opublicerat manuskript av Knut Wicksell (An Unpublished Manuscript by Knut Wicksell), with a foreword by T. Hedlund-Nyström, Lars Jonung and Bo Sandelin. This manuscript with extended comments on the forestry part, is published in English as Hedlund-Nyström et al (1993).

<sup>4</sup> See Wicksell (1911).

<sup>5</sup> For an account of some of the errors, see e.g. Samuelson (1976), Löfgren (1983) and Löfgren (1993).

## THE "FAUSTMANN SOLUTION"

As is well known today, it was Martin Faustmann who, in 1849, formulated the optimal rotation problem correctly, but it was Max Robert Pressler (1860) who by introducing the concept "das Weiserprozent", first came up with a correct first order condition for the optimal rotation period. The solution of the rotation problem is well defined provided that the following conditions are met:

- (i) The capital market is perfect in the sense that one can lend and borrow any amount at the prevailing interest rate  $\bar{p}$ , and the interest rates are known for the future.
- (ii) The timber price is known for the future.
- (iii) Forest land can be bought and sold in a perfect market.
- (iv) The technical lumber yields are known for the future.

Let:

$pf(t)$  = the value of the forest stand at time  $t$ .

$f(t)$  = the stock of timber at time  $t$ .

$p$  = the price of timber.

$\bar{p}$  = the constant discount rate.

The rotation problem can now be formulated as the maximization of the present value of the income stream with respect to  $t$ :

$$\begin{aligned} \text{Max}_t \left\{ pf(t)e^{-\bar{p}t} (1 + e^{-\bar{p}t} + e^{-\bar{p}2t} \dots) \right\} = \\ \text{Max}_t \left\{ \frac{pf(t)}{e^{\bar{p}t} - 1} \right\} = V(p; \bar{p}) \end{aligned} \quad (1)$$

The first-order conditions for an interior solution to (1) can be written in the following manner<sup>6</sup>:

<sup>6</sup> When we are disregarding planting costs the optimal rotation period will be independent of  $p$ . Das Weiserprozent  $i^*$ , is defined as:

$$i^* = \frac{f'(t)}{f(t)} - \frac{\bar{p}}{e^{\bar{p}t} - 1}$$

and it is optimal to cut the stand when  $i^* \leq \bar{p}$ .

$$f'(t) = \bar{\rho}f(t) + \frac{\bar{\rho}f(t)}{e^{\bar{\rho}t} - 1} \quad (2)$$

A verbal interpretation gives:

***The Faustmann-Pressler<sup>7</sup> theorem:** A forest stand shall be cut down, when the time rate of change of its value is equal to interest on the value of the stand plus interest on the value of the forest land.*

There is, however, an alternative way of deriving the same theorem. Instead of demanding the maximum present value, one could instead demand the maximum annual rent that a perfect market would generate for the privilege of being allowed to use the land for forestry.

Formally, the rent  $r$  emerges as the solution of the problem:

$$\text{Max}_t r \quad (3)$$

Subject to ( $p$  is put equal to one):

$$f(t)e^{-\bar{\rho}t} - r \int_0^t e^{-\bar{\rho}s} ds = 0 \quad (4)$$

Solving the constraint for  $r$  yields:

$$r = \frac{\bar{\rho}f(t)}{e^{\bar{\rho}t} - 1} \quad (5)$$

Maximizing (5) with respect to  $t$  yields the same rotation period as the one which emerges from equation (2). The resulting rent,  $r^*$ , has the interpretation as interest on capital, or Lindahl-Hicks income.

<sup>7</sup> Elsewhere (Johansson & Löfgren, 1985; and Löfgren, 1983) I have named the theorem The Faustmann-Pressler-Ohlin theorem to acknowledge that Bertil Ohlin at the age of 18 derived it. His paper was published in 1921 in a special issue of Ekonomisk Tidskrift (today The Scandinavian Journal of Economics) to honour Wickseil on his 70:th birthday. An English translation of this paper entitled "Concerning the Question of the Rotation Period in Forestry" can be found in Journal of Forest Economics, 1:1, 1995.

## WICKSELL ON WINE AGING

Inspired by Jevons (1876), Wicksell defines the value of wine as a function of its age,  $g(t)$ , and he assumes that the price of grape juice is exogenously given by  $\bar{p}_0$ . He, however, treats the interest rate  $\rho$ , the "yield energy", as a variable to be maximized subject to a nonarbitrage condition that the price of grape juice equals the discounted value of the value of wine of age  $t$ , or:

$$g(t)e^{-\rho t} = \bar{p}_0 \quad (6)$$

The optimization problem of the wine owner can now, after solving (6) for  $\rho$ , be written:

$$\text{Max}_t \rho = \frac{1}{t} [\log g(t) - \log \bar{p}_0] \quad (7)$$

The first order condition for maximum is:

$$\frac{g'(t)}{g(t)} - \frac{1}{t} [\log g(t) - \log \bar{p}_0] = 0 \quad (8)$$

or if we like:

$$\frac{g'(t)}{g(t)} = \rho^*(t; \bar{p}_0) \quad (9)$$

The interpretation is that the optimal yield energy equals, as Jevons puts it, "the rate of increase of the produce divided by the whole produce", which is often referred to as Jevon's interest formula.

In the perfect capital market version of wine aging solution we would take the interest rate as exogenously given,  $\bar{\rho}$ , and choose  $t$  to maximize the value of grape juice:

$$\text{Max}_t p_0 = \text{Max}_t \{g(t)e^{-\bar{\rho}t}\} \quad (10)$$

The first order condition is obviously:

$$\frac{g'(t)}{g(t)} = \bar{\rho} \quad (11)$$

Moreover, if we put  $\bar{p}_0$  in (7) equal to the solution of (10), we would obtain  $\rho^* = \bar{\rho}$  and the same optimal rotation period as generated by (11).

#### WICKSELL ON FORESTRY

Given his treatment of wine aging, how would Wicksell treat forestry? He would certainly treat the yield energy as an endogenous variable to be maximized, and he would regard the yearly cost to rent the forest land as exogenously given,  $r = \bar{r}$ , but subject to a nonarbitrage condition. More formally the objective of the firm renting the forest land would be:

$$\text{Max}_t \rho \quad (12)$$

subject to:

$$\bar{r} = \frac{\rho f(t)}{e^{\rho t} - 1} \quad (13)$$

i.e. maximize the yield energy subject to the condition that the exogenously given land rent equals interest on the value of land. In this case we cannot solve (13) explicitly for  $\rho$ , but we can use the constraint to establish the existence of an implicit function<sup>8</sup>  $\rho(t; \bar{r})$ . At the optimum it must hold that:

$$\frac{\partial \rho}{\partial t} = 0 \quad (14)$$

Using the implicit function theorem once again, differentiating (13) with respect to  $t$ , and solving when  $\frac{\partial \rho}{\partial t} = 0$ ,

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<sup>8</sup> Another possibility would be to use a Lagrangian.



we obtain:

$$f'(t) = \rho(t, \bar{r})f(t) + \frac{\rho(t; \bar{r})f(t)}{e^{\rho(t; \bar{r})t} - 1} \quad (15)$$

Or, as Wicksell puts it in his "unpublished" manuscript: "One of the conditions for the maximum of  $\rho$  is":

$$\rho^* = \frac{f'(t) - \bar{r}}{f(t)} \quad (16)$$

This equation, which is equation (2) in Wicksell's manuscript, first confused me, but by substituting for  $\bar{r}$  from equation (13) one obtains equation (15).

In the same spirit as in the wine aging problem we can put  $\bar{r} = r^*$ , and solve (12) subject to (13) and obtain  $\rho^* = \bar{\rho}$ , and the same rotation period as in the Faustmann problem.

#### CONCLUDING COMMENTS

Today a perfect capital market approach to forestry is more natural, but when Wicksell wrote his paper (before 1902, see Hedlund-Nyström et al., 1993) a key problem was to understand the nature of interest, capital, and distribution. In such a case the idea to use a forest (or wine aging) as an example was a good idea. As Wicksell puts it in his Lectures introducing wine aging: "The following alternative construction is introduced to let time, which is the core of the concept of capital appear in the most pure way"<sup>9</sup>. One can, like Harry G. Johnson (1973), claim that Wicksell's

<sup>9</sup> Author's translation. The dating of the paper is based on some technicalities with respect to the use of compound interest (a continuous discount factor) instead of simple interest. These details moves the point of time beyond 1898 and Geldzins und Güterpreise. The reason why it is believed to be written before 1902 is that Wicksell claims that the fact that the sum of wage and capital income exhausts production implies that the production function is homogenous of degree one. In 1902 he published a paper in *Ekonomisk Tidskrift* where he notes that this condition would also be fulfilled at average cost minimum, where the average productivity equals the marginal productivity. My personal bet is that it was written autumn 1900 or spring 1901. This guess is based on the fact that Wicksell's lectures at Lund during the spring semester 1901 had agriculture and forestry as main topics; see Gårdlund (1958).

choice of wine ageing is a bit unsatisfactory in this respect, because the prices paid for older wine are dependent on tastes, but presumably also influenced by relative quantities of the different vintages produced. When Johnson moves from there to a model of forestry without considering the rotation element and the resulting scarcity of land he is, as we have seen, inadvertently misrepresenting Wicksell's thinking on forestry.

Wicksell's treatment of forestry is also, as far as I can see, obviously independent of the early German contributions, both in terms of general approach and notation. The "correct" solution to the rotation problem was in principle known to Swedish foresters already in 1876 through C.G. Holmerz's book "Studier i skogstaxation", del 1 (Studies in Forest Surveying, part 1), but as the discussion in Löfgren (1990) about the Swedish profitability war (1907–1917) shows, neither foresters nor an economist like Eli F. Heckscher had any fundamental insights. It is certainly interesting to speculate about the development of the discussion of the determination of the optimal rotation period during the profitability war, if Wicksell had chosen to deal with trees rather than wine in his "Lectures". My guess is that it would have changed very little. Wicksell was difficult to understand even for a mathematical economist like Ragnar Frisch, and the participants of the profitability war in Swedish forestry were not even close to Frisch's competence in economic theory.

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