



A NOTE ON THE TREE-CUTTING PROBLEM IN A STOCHASTIC ENVIRONMENT

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ABSTRACT

The optimal harvest rule in a stochastic world, proposed by Clarke and Reed, ignored the additional rental and management costs incurred if a harvest is postponed. Noting the critical role these costs play in any practical decision-making process, we modify the Clarke-Reed rule and derive a set of comparable but more meaningful analytical results. We further argue that there no longer exist stochastic counterparts of the Wicksellian and Faustmann "tree-cutting" problems once land rent and management expenditure are explicitly included in the harvest rule formulation.

Keywords: Optimal harvesting rule, stochastic processes, waiting cost.

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INTRODUCTION

In recent years forest economics has witnessed a resurgent research interest in the Wicksellian and Faustmann "tree-cutting" problems in a stochastic environment. Clarke and Reed (hereafter, CR; 1989) represents one of the important attempts in the field by distinguishing price and growth uncertainty and validating the so-called myopic-look-ahead (MLA) harvest rule. They derived a set of analytical results for their problem although no arbitrary boundary conditions were employed. Their results are not only consistent with, but also expand previous results (see, e.g., Norstrom, 1975; Brock, Rothschild & Stiglitz, 1979, 1989; Malliaris & Brock, 1982). Thus, it can be said that CR's study improved our knowledge of the tree-cutting problem in a more realistic world.

However, like most of the early works on the traditional Wicksellian "tree-cutting" problem (single rotation) in a stochastic world, one critical shortcoming of CR's study is that waiting was assumed to be costless. This is clearly reflected in the stopping rule they proposed — "that the op-

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timal harvest time should be characterized by the condition that the asset's intrinsic value at τ should just equal its discounted expected intrinsic value at an infinitesimal time later, $\tau + d\tau$ " (Clarke & Reed, 1989, p. 576). As Samuelson (1976, p. 471) correctly pointed out (though he was concerned with cutting rules in a deterministic world), "maximizing the present discounted value, over one planting cycle..., will give you a somewhat too long rotation period and will not enable you to cover the land rent that will be set by your more perspicacious competitors." If a tree is not cut now, it must occupy the land and continue to be managed; therefore the decision-maker will incur rental and management costs by delaying the cutting. Without taking these costs into consideration, any timing and valuation rule could be misleading and incomplete.

It should be noted that there is exception to the this Wicksellian-type stopping problem. Miller and Voltaire (hereafter, MV; 1980, 1983), following the suggestion made and techniques developed by Brock, Rothschild and Stiglitz, established a formula for the repeated (infinite rotations) tree-cutting problem assuming that a tree's size (and value) behaves as a diffusion process. This is the Faustmann counterpart in a stochastic world. In this fashion, land rent is implicitly included in the maximization formulation. However, it is difficult to imagine a situation where the parameters of the diffusion process followed by the tree's size (and value) — the drift and instantaneous variance — will remain unchanged throughout time such that repeated uniform rotations can be maintained.

To overcome this problem and to take the land rent explicitly into account, one alternative to the MV formulation is, as proved by Samuelson (1976, p. 479), to adopt the stopping period "that results from maximizing the present discounted value net algebraic receipts over the first cycle, but with the market land rental included in those receipts". In so doing, the maximization problem of infinite like rotations with a strong assumption on the stochastic process is literally reduced to one of a single rotation with a more acceptable assumption.

Moreover, it should be recognized that in addition to the land rent, there are certain *recurring* management costs, such as fire prevention, pest control, road maintenance and

oversight, associated with any decision to postpone harvest. These costs must also be treated in the analysis (Haight & Holmes, 1991). This short note is specifically aimed at modifying CR and MV's studies by including these costs of waiting into the model formulation.¹ It will be seen that because of this modification, the optimal timing and valuation rules and attendant analytical results become, though more complicated, more sensible than were claimed before.

MODEL SPECIFICATION

Following CR's notation, suppose a biological asset (a stand of trees) is subject to random variability in growth for which unit price is exogenous and random as well. The valuation procedure and harvest policy of the asset can be described in terms of the following variables:

- $P(t)$ = unit price of the asset at t ,
- $q(t) = \log P(t)$,
- $X(t)$ = asset size at t ,
- $y(t) = \log X(t)$,
- $R(t) = P(t) X(t)$, the gross intrinsic value (revenue) at t ,
- $g(t)$ = the deterministic component of the asset's instantaneous proportional growth rate at age $A(t) = t$ (assuming $A(t) = 0$ at time $t = 0$),
- δ = positive constant discount rate.

To incorporate the rental and management costs mentioned above, we simply add that

- $C(t)$ = additional rental and management costs incurred at t , which are deterministic.²

Therefore, the asset's net market value at t is defined as $W(t) \equiv W(t, y, q, C)$, and asset size and price evolve as the following stochastic processes:

$$dq = b dt + \sigma_q dw_q \quad (1)$$

$$dy = g(t) dt + \sigma_y dw_y \quad (2)$$

¹ Indeed, there is also an option cost associated with waiting (see, e.g., McDonald & Siegel, 1986). But for simplicity, it is ignored here.

² This is for ease of exposition. Of course, we can assume that these costs are also stochastic. The resulting analytics will be similar.

That is, the asset's unit price behaves as a geometric Brownian motion with constant drift b , and constant variance σ_q^2 . The logarithm of asset size is a diffusion that behaves locally like a Brownian motion with drift $g(t)$ and constant variance σ_y^2 ; w_q , and w_y are standard Wiener processes.³ Note that we assume that the growth process is *age dependent* and the proportionate growth rate, $g(t)$, is decreasing and differentiable in t .⁴

OPTIMAL HARVEST RULE

Based on above definitions and specifications, we can reformulate the asset valuation procedure. What should be emphasized is that when rental and management costs are incorporated, the asset's owner maximizes the *net* market value of the asset not its *gross* market value. Assuming risk neutrality, at time t , the asset's net market value $W(t) \equiv W(t, y, q, C)$ is its net expected present value assuming that it is harvested at its net-expected-present-value-maximizing harvest date:

$$\begin{aligned} W(t) &= W(t, y, q, C) \\ &= \sup_{\tau \geq t} E_t \left\{ \exp[-\delta(\tau - t) + q(\tau) + y(\tau)] - \exp[-\delta(\tau - t)] \right. \\ &\quad \left. \int_0^\tau C(s) \exp(\delta s) ds \mid y(t) = y, q(t) = q \right\} \end{aligned} \quad (3)$$

where E_t denotes the mathematical expectation, conditional on values at time t , and s is the factor of integration.

The problem of determining a closed-form expression for (3), given the dynamics of (1) and (2), is simplified once the optimal stopping rule, prescribing the conditions under which the asset should be harvested, is determined.

³ Clarke & Reed (1989) extensively discussed alternative specifications of the asset's unit price and size processes and their implications. We maintain their specification given the fact that we believe that their arguments for it are valid.

⁴ See Reed & Clarke (1990) for the case of *pure size dependent* growth process.

Here we modify the "myopic-look-ahead" (MLA) rule proposed by CR. Instead of judging the evaluation in gross terms, our new rule states that the optimal harvest time (in a continuous time framework) should be characterized by the condition that the asset's *net* intrinsic value

$$R(\tau) - \int_0^\tau C(t) \exp(\delta t) dt \text{ at } \tau$$

should just equal its discounted *net* expected intrinsic value at an infinitesimal time later $t + d\tau$. That is,

$$\begin{aligned} \exp[y(t) + q(\tau)] = \\ E_\tau \left\{ \exp(-\delta d\tau) \left[\exp(y(\tau) + dy + q(\tau) + dq) - C(\tau) d\tau \right] \right\} \end{aligned} \quad (4)$$

which implies

$$\begin{aligned} \exp(y + q) = \\ E_\tau \left\{ \exp(-\delta d\tau) \left[\exp(y + q) \exp(dy + dq) - C(\tau) d\tau \right] \right\} \end{aligned} \quad (5)$$

or

$$\begin{aligned} 1 = E_\tau \left\{ \exp(-\delta d\tau) \left[\exp(dy + dq) - \frac{C(\tau) d\tau}{\exp(y + q)} \right] \right\} \\ = (1 - \delta d\tau) E_\tau \left\{ 1 + dy + dq + \frac{1}{2} (dy + dq)^2 - \frac{C(\tau) d\tau}{R(\tau)} + o(d\tau) \right\} \end{aligned} \quad (6)$$

where $o(d\tau)$ collects terms that go to zero faster than $d\tau$. Using (1), (2) and the fact that $E(dw^2) = d\tau$ for a Wiener process we have, in the limit as $d\tau \rightarrow 0$,

$$g(\tau) = \delta - b - \frac{1}{2} (\sigma_y^2 + 2\sigma_{yq} + \sigma_q^2) + \frac{C(\tau)}{R(\tau)} \quad (7)$$

Notice that in deriving this result, we assume that there exists a correlation between the disturbances in the y and q processes.

With pure age-dependent growth, the MLA rule implies harvesting at some fixed age (say, τ_M) where nonrandom component of the asset's proportional growth equals the right hand side of (7). Clearly, with this MLA rule the size at which the asset is harvested becomes a random variable, depending on the expected rates of interest, price and cost changes and the instantaneous variances of all stochastic processes. The inclusion of waiting costs makes the decision different from that described by CR in their equation (5) in that the trees must be harvested at an earlier time if these costs are to be compensated. Further, it can be seen that if the sum of the instantaneous changes in asset size and price is greater than the sum of interest rate plus the rate of cost change relative to the gross value of the asset, then the asset is still appreciating; therefore, its harvest should be postponed. Conversely, if the sum of the instantaneous changes in asset size and price becomes smaller than the sum of interest rate plus the rate of cost change relative to the gross value of the asset, then the asset is depreciating and it should have been harvested earlier.

It thus follows that the net asset value at $t < \tau_M$ is

$$\begin{aligned}
 W^M(t) &= W^M(t, y, q, C) \\
 &= E_t \left\{ \exp[-\delta(\tau_M - t) + q(\tau_M) + y(\tau_M)] \right. \\
 &\quad \left. - \left[\int_0^t C(s) \exp(\delta s) ds + \exp(-\delta(\tau_M - t)) \int_t^{\tau_M} C(s) ds \right] \right\} \\
 &= e^{y+q} \exp \left[\int_t^{\tau_M} \left(g(s) + b - \delta + \frac{1}{2} (\sigma_y^2 + 2\sigma_{yq} + \sigma_q^2) \right) ds \right] \\
 &\quad - \int_0^t C(s) \exp(\delta s) ds - \exp(-\delta(\tau_M - t)) \int_t^{\tau_M} C(s) ds \} \quad (8)
 \end{aligned}$$

Now the question is whether the MLA harvesting rule at age τ_M is still optimal after our modification. From the results of Krylov (1980) and Shirayayev (1978), we know that the stopping rule, harvesting at τ_M , will be an optimal harvest strategy provided certain regularity conditions are satisfied, and the value function satisfies (a) the Hamilton-

Jacobi-Bellman equation

$$\delta W(t, z, C) = [g(t) + b]W_z + \frac{1}{2}\sigma_z^2 W_{zz} + W_t,$$

where subscripts represent partial derivatives;⁵ (b) the condition that $W^M > W^I$ (W^I is the net intrinsic value of the asset not its market value) on the continuation region $t > t_M$; and (c) the following continuity and smooth-pasting conditions at the stopping boundary $t = t_M$:

$$W(\tau_M, z, C) = \exp(z) - \int_0^{\tau_M} C(s) \exp(\delta s) ds,$$

$$W_z(\tau_M, z, C) = \exp(z),$$

$$W_t(\tau_M, z, C) = \left(\delta - g(t) - b - \frac{1}{2}\sigma_z^2\right) \exp(z) - \delta \int_0^{\tau_M} C(s) \exp(\delta s) ds.$$

The H-J-B equation holds for $t < \tau_M$ since for $W = W^M$, using (8),

$$\begin{aligned} [g(t) + b]W_z + \frac{1}{2}\sigma_z^2 W_{zz} + W_t = \\ [g(t) + b]\exp(z) + \frac{1}{2}\sigma_z^2 \exp(z) + \left[\delta - g(t) - b - \frac{1}{2}\sigma_z^2\right]\exp(z) \\ - \delta \int_0^{\tau_M} C(s) \exp(\delta s) ds = \delta W^M \end{aligned}$$

as required. The condition that $W^M > W^I$ for $t < \tau_M$ follows from (8) and the assumption that $g(t)$ is decreasing and satisfies (7). The continuity and smooth-passing conditions can be easily verified as well. Thus, the MLA rule is optimal and requires that the asset be harvested at age τ_M .

⁵ For notational convenience, we redefine $z \equiv y + q = \log R(t)$. Therefore,

$$dz = [g(t) + b]dt + \sigma_z dw_z, \quad \sigma_z^2 = \sigma_y^2 + \sigma_q^2 + 2\sigma_{yq}.$$

COMPARATIVE STATICS

We now examine the question of how the optimal harvest age and the preharvest market valuations of the asset respond to the changes in the parameters of the problem. Using equations (7), (8) and letting the correlation coefficient of price and growth processes $\rho = \sigma_{yq}/\sigma_y\sigma_q = 0$, we find

$$\frac{\partial \tau_M}{\partial \sigma_q} = -\frac{\sigma_q}{g'(\tau_M)} > 0,$$

$$\frac{\partial \tau_M}{\partial \sigma_y} = -\frac{\sigma_y}{g'(\tau_M)} > 0,$$

$$\frac{\partial \tau_M}{\partial C} = \frac{1}{Rg'(\tau_M)} < 0,$$

$$\frac{\partial \tau_M}{\partial \delta} = \frac{1}{g'(\tau_M)} < 0,$$

$$\frac{\partial W^M}{\partial \sigma_q} = (\tau_M - t)\sigma_q \exp(z) \exp\left(\int_t^{\tau_M} dz\right) > 0,$$

$$\frac{\partial W^M}{\partial \sigma_y} = (\tau_M - t)\sigma_y \exp(z) \exp\left(\int_t^{\tau_M} dz\right) > 0,$$

$$\frac{\partial W^M}{\partial b} = \frac{\partial W^M}{\partial g} = (\tau_M - t) \exp(z) \exp\left(\int_t^{\tau_M} dz\right) > 0,$$

$$\frac{\partial W^M}{\partial C} = -(\tau_M - t) \int_t^{\tau_M} \exp(\delta s) ds < 0,$$

$$\frac{\partial W^M}{\partial \delta} = -(\tau_M - t) \left[\exp(z) \exp\left(\int_t^{\tau_M} dz\right) - \int_t^{\tau_M} C(s) \exp(-\delta s) ds \right] < 0.$$

To sum up, stochastic variations in both price and growth processes tend to increase the optimum harvesting age and the current market value of the forest asset. Decreases in

price and growth drifts, increases in additional rental and management costs, and increases in the discount rate decrease the optimum harvesting age and the current market value of the asset. These results confirm to our intuition and are consistent with the existing empirical evidence (Lomander, 1987; Brazee & Mendelsohn, 1988) as well as CR's findings.

CONCLUSION

This paper has extended the stochastic "tree-cutting" analysis from valuation of gross present revenues to valuation of net present revenues. It is argued that postponing timber harvest inevitably entails waiting costs in such items as land occupation, fire prevention, road maintenance and oversight. These costs are nontrivial and should be reflected in any meaningful tree-cutting rule. When they are incorporated into the timing and valuation formulas, we find that the optimal stopping time must come earlier than that suggested by CR, so that the opportunity loss of delaying harvest can be offset with a higher growth rate. Therefore, the prescribed rotation length becomes shorter, although most of CR's comparative static results are maintained. In addition, we believe that once the waiting costs are considered, it becomes unnecessary to work on the infinite rotation problem. As such, the distinction between the so-called Wicksellian wine aging problem and Faustmann tree-cutting regime becomes redundant. In short, CR's work has contributed to our understanding of how to apply stochastic investment theory to overcome some of the shortcomings in conventional deterministic timing and valuation techniques, but the absence of waiting costs in their model is a critical omission. In this note, we have raised our concern and attempted to modify their model accordingly.

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